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On Two Polynomial Inequalities of Erdős Related to Those of the Brothers Markov

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Polynomials of degree at most n which are real on the real axis and do not vanish in the open unit disk are considered. Sharp point-wise bounds for the derivative $p'(x)$ at an arbitrarily prescribed point x_0 of the unit interval, in terms of the maximum of $|p(x)|$ on $[-1, 1]$ are obtained. Certain other related problems are also solved. © 1996 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. *Inequalities of the Brothers Markov*

Let $\mathscr{P}_n$ denote the class of all polynomials $p(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree at most n. Motivated by a question asked by the chemist Mendeliev, Markov [10] proved the following.

THEOREM A. *If $p \in \mathscr{P}_n$ and $|p(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$, then*

$$|p'(x)| \leqslant n^2 \qquad \text{for} \quad -1 \leqslant x \leqslant 1. \tag{1}$$

In (1), equality is possible only at -1, $+1$ and holds only for $p(x) = e^{i\gamma} T_n(x)$, $\gamma \in \mathbb{R}$, where $T_n(x) = \cos(n \arccos x)$ is the nth Chebyshev polynomial of the first kind. This led W. Markov ([11]; much later Bernstein arranged to get a German version of the paper published in [12] so as to make the work more accessible) to look for the best possible estimate for $|p^{(k)}(x^*)|$ at any prescribed point $x^* \in [-1, 1]$ if $p \in \mathscr{P}_n$ and $|p(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$.

Given k $(1 \leqslant k \leqslant n)$, and $x^* \in [-1, 1]$ let p_* be an extremal polynomial; i.e.,

$$|p_*^{(k)}(x^*)| = \sup\{|p^{(k)}(x^*)| : p \in \mathscr{P}_n \text{ and } \max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant 1\}.$$

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It is easily seen that such a polynomial exists and that $\max_{-1 \leqslant x \leqslant 1} |p_*(x)| = 1$. W. Markov was able to characterize and even identify the extremal polynomials for different values of x^*. Let $\xi_1 < \xi_2 < \cdots < \xi_{n-k}$ and $\eta_1 < \eta_2 < \cdots < \eta_{n-k}$ be the zeros of $(x+1)\,T_n^{(k+1)}(x) + kT_n^{(k)}(x)$ and of $(x-1)\,T_n^{(k+1)}(x) + kT_n^{(k)}(x)$, respectively. They all lie in $(-1, 1)$ and interlace, i.e., $-1 < \xi_1 < \eta_1 < \xi_2 < \cdots < \xi_{n-k} < \eta_{n-k} < 1$. W. Markov showed that the polynomial T_n is extremal for x^* belonging to any of the $n-k+1$ intervals

$$[-1, \xi_1], [\eta_1, \xi_2], ..., [\eta_{n-k-1}, \xi_{n-k}], [\eta_{n-k}, 1].$$

Besides, he noted that the points $\lambda_j = (\sec^2(\pi/n))\,\xi_j + \tan^2(\pi/2n)$ lie in (ξ_j, η_j) for $j = 1, ..., n-k$. He showed that at a point x^* belonging to $(\xi_j, \lambda_j]$ written as

$$x^* := \left(\frac{c+1}{2}\right)\xi_j + \frac{c-1}{2}, \qquad 1 < c \leqslant 1 + 2\tan^2\frac{\pi}{2n},$$

the polynomial

$$T_n\left(\frac{2x+1-c}{c+1}\right) \equiv T_n\left(\frac{(1+\xi_j)(x-x^*)}{1+x^*} + \xi_j\right)$$

is extremal. The point $\mu_j := (\sec^2(\pi/2n))\,\eta_j - \tan^2(\pi/2n)$ also lies in (ξ_j, η_j) for $j = 1, ..., n-k$ and at a point $x^* \in [\mu_j, \eta_j)$ the polynomial

$$T_n\left(\frac{(1-\eta_j)(x-x^*)}{1-x^*} + \eta_j\right) \qquad (j = 1, ..., n-k)$$

is extremal. There remain the intervals (λ_j, μ_j), $j = 1, ..., n-k$. The equation

$$\frac{d^k}{dx^k}\left((x^2-1)\,T'_{n-1}(x)\right) = 0$$

has a root v_j in (λ_j, μ_j) for each $j \in \{1, ..., n-k\}$. The polynomial T_{n-1} is extremal when x^* coincides with any of the $n-k$ points $v_1, ..., v_{n-k}$. Finally, for x^* belonging to the intervals (λ_j, v_j) and (v_j, μ_j), the extremal polynomial satisfies a differential equation of the form

$$1 - (p(x))^2 = \frac{(1-x^2)(x-b)(x-c)}{n^2(x-a)^2}\,(p'(x))^2.$$

Here a, b, and c are real constants which depend on a parameter. In particular, he proved the following.

COROLLARY A. *If* $p(x)=\sum_{\nu=0}^{n} a_\nu x^\nu$ *satisfies the conditions of Theorem* A, *then for* $1 \leqslant k \leqslant n$ *we have*

$$|a_k| = \frac{1}{k!}\,|p^{(k)}(0)| \leqslant \begin{cases} \dfrac{1}{k!}\,|T_n^{(k)}(0)| & \text{if } \; n-k \text{ is even} \\ \dfrac{1}{k!}\,|T_{n-1}^{(k)}(0)| & \text{if } \; n-k \text{ is odd,} \end{cases} \tag{2}$$

$$\max_{-1 \leqslant x \leqslant 1} |p^{(k)}(x)| \leqslant T_n^{(k)}(1). \tag{3}$$

An alternative approach to the problem considered by W. Markov was developed by Voronovskaja in a long series of papers. The reader will find a comprehensive account of her work in [18]. For a detailed discussion of various extensions and generalizations of the (polynomial) inequalities of the brothers Markov see [15, 16]. Numerous papers have been written on the topic but [7, 13, 9, 4] are amongst those containing some of the most striking contributions.

Remark 1. It was shown by Bernstein [1] that if p satisfies the conditions of Theorem A, then

$$|p'(x)| \leqslant \frac{n}{\sqrt{1-x^2}} \qquad \text{for} \quad -1 < x < 1. \tag{4}$$

However, this well-known result which is certainly very elegant gives the sharp bound for $|p'(x)|$ only when $x = \cos((2\nu-1)\,\pi/2n)$ $(\nu = 1, ..., n)$.

It is also of interest to estimate $|p(x^*)|$ at a point $x^* \in \mathbb{R} \setminus [-1, 1]$ but that was already done by Chebyshev (see [3, p. 7]). He showed that

$$|p(x^*)| \leqslant |T_n(x^*)| \qquad \text{for} \quad x^* \in \mathbb{R} \setminus [-1, 1]. \tag{5}$$

1.2. *The Results of Erdős*

The extremal polynomials p_* defined and briefly discussed above have at most one zero outside the interval $[-1, 1]$. As such, it should be possible to improve upon all the preceding inequalities if the zeros of p are required to stay away from $[-1, 1]$. As regards (1), Erdős [8; see in particular, the second half of p. 311] proved the following.

THEOREM B. *Let* $p \in \mathscr{P}_n$ *be such that* $|p(x)| \leqslant 1$ *for* $-1 \leqslant x \leqslant 1$. *If the zeros of* p *are all real and lie on* $\mathbb{R} \setminus (-1, 1)$, *then*

$$|p'(x)| \leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right)^{-n+1} \cdot n \qquad \textit{for} \quad -1 \leqslant x \leqslant 1. \tag{6}$$

In (6) equality is possible only at -1, $+1$ and holds for

$$p(x) := e^{i\gamma} \frac{n^n}{2^n(n-1)^{n-1}} (1+x)(1-x)^{n-1}, \qquad \gamma \in \mathbb{R},$$

$$p(x) := e^{i\gamma} \frac{n^n}{2^n(n-1)^{n-1}} (1+x)^{n-1} (1-x), \qquad \gamma \in \mathbb{R},$$

respectively.

The first such result providing an estimate for $|p'(x)|$ at a given point $x \in (-1, 1)$ was also proved by Erdős [8]. It is to be compared with (4) and reads as follows.

THEOREM C. *Let $p \in \mathscr{P}_n$ and suppose that $|p(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$. If $p(x)$ is real for real x and $p(z) \neq 0$ for $|z| < 1$, then*

$$|p'(x)| \leqslant 4\sqrt{n}/(1-|x|)^2 \qquad \textit{for} \quad -1 \leqslant x \leqslant 1. \tag{7}$$

Here $\sqrt{n}$ cannot be replaced by any function of n tending to infinity more slowly but the bound $4\sqrt{n}/(1-|x|)^2$ is far from being best possible for any x. The following result was presented in [14].

THEOREM D. *Let f be a rational function of degree n having neither zeros nor poles in $|z| < 1$. If $f(x)$ is real for $x \in (-1, 1)$ and $|f(x)| \leqslant 1$ for $-1 < x < 1$, then*

$$|f'(x)| \leqslant \sqrt{(2/e)}\sqrt{n}/(1-x^2) \qquad \textit{for} \quad -1 < x < 1. \tag{8}$$

For p satisfying the conditions of Theorem C, we have

$$|p'(x)| \leqslant \sqrt{(1/e)}\sqrt{n}/(1-x^2) \qquad \textit{for} \quad -1 < x < 1. \tag{9}$$

The estimate (8) was shows to be *best possible* for each x in $(-1, 1)$ in the sense that for any given ξ in $(-1, 1)$ there exists a rational function $f_{n,\xi}$ satisfying the conditions of Theorem D for which

$$(1-\xi^2)\,|f'_{n,\xi}(\xi)| > \sqrt{(2/e)}\sqrt{n} - O(1/\sqrt{n}) \qquad \text{as} \quad n \to \infty.$$

The same cannot be said about (9) *except* when $x = 0$. In that case it says that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a real polynomial of degree at most n not vanishing in $|z| < 1$ such that $|p(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$, then

$$|a_1| = |p'(0)| \leqslant \sqrt{(1/e)}\sqrt{n}. \tag{10}$$

It was shown in [14] that at least when $\sqrt{n} \in \mathbb{N}$ there exists a polynomial p of degree n satisfying the above conditions with

$$|p'(0)| = \sqrt{(1/e)}\,\sqrt{n} - \frac{1}{12\sqrt{e}}\,\frac{1}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right).$$

For p satisfying the conditions of Theorem C and arbitrary $x^* \in [-1, 1]$ we obtain in this paper sharp estimate for $|p'(x^*)|$ in the spirit of W. Markov's result, which will allow us, in particular, to replace (10) by an inequality that cannot be improved for any n belonging to $\mathbb{N}$ (see Corollary 2). We would have liked to do the same for $|p^{(k)}(x^*)|$, where k is an arbitrary positive integer $\leqslant n$ but for $k \in \{2, ..., n-1\}$ the problem seems to be much harder. Besides, we obtain the exact bound for $|p(x^*)|$ at an arbitrary point $x^* \in \mathbb{R} \setminus [-1, 1]$, which is a result corresponding to Chebyshev's inequality (5).

1.3. *The Class* π_n

We shall denote by $\mathscr{P}_n^+$ the subclass of polynomials $p \in \mathscr{P}_n$ which do not vanish in the open unit disk and take positive values on $(-1, 1)$. Note that if p satisfies the conditions of Theorem C then p or $-p$ belongs to $\mathscr{P}_n^+$.

For $k = 0, 1, ..., n$ let

$$\begin{aligned} q_{n,k}(x) &:= (1+x)^k\,(1-x)^{n-k}, \\ q_{n,k,*}(x) &:= \frac{n^n}{2^n k^k (n-k)^{n-k}}\,(1+x)^k\,(1-x)^{n-k}, \end{aligned} \tag{11}$$

and denote by π_n the subclass of polynomials $p \in \mathscr{P}_n$ which can be expressed as $\sum_{k=0}^{n} A_k (1+x)^k\,(1-x)^{n-k}$, where $A_k \geqslant 0$ for $k = 0, 1, ..., n$.

At the end of Volume II of d'Adhémar's "Principes d'Analyse" there appears a note, written by S. Bernstein (also see his Doctoral dissertation "On the Best Approximations of Continuous Functions," Sections 58–60) formulating a minimization problem for which he had to transform an arbitrary polynomial $p(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ into the form $\sum_{k=0}^{m} A_k (1+x)^k\,(1-x)^{m-k}$, where $m \geqslant n$. Among other things he showed that a *real* polynomial $p \in \mathscr{P}_n$ does not change sign on $(-1, 1)$ if and only if, for sufficiently large m, it can be written as a sum $\sum_{k=0}^{m} A_k (1+x)^k\,(1-x)^{m-k}$ whose coefficients A_k have the same sign (also see [2; 3, p. 47]). The number m can be taken to be n if in particular $p(z) \neq 0$ in $|z| < 1$, i.e., $\mathscr{P}_n^+ \subseteq \pi_n$. This latter fact was proved in [17, p. 355], where the following "extension" of Theorem B was obtained.

Theorem B′. *If* $p \in \pi_n$ *then*

$$|p'(x)| \leqslant \tfrac{1}{2} e n \max_{-1 \leqslant x \leqslant 1} p(x) \qquad \textit{for} \quad -1 \leqslant x \leqslant 1. \tag{12}$$

1.3.1. *The Polynomials* $P_{n,a}$. We need to introduce for each $a \in (0, 1]$ the polynomial

$$P_{n,a}(x) := a\left(\frac{1+x}{2}\right)^n + \lambda_a \frac{n^n}{2^2(n-2)^{n-2}}\left(\frac{1+x}{2}\right)^{n-2}\left(\frac{1-x}{2}\right)^2, \tag{13}$$

where λ_a is the largest number such that $\max_{-1 \leqslant x \leqslant 1} P_{n,a}(x) = 1$. This polynomial which belongs to π_n plays an important role in our work. Clearly, there is one and only one point τ_a in $[-1, 1)$, where $P_{n,a}$ assumes the value 1. It may be noted that $P_{n,a}$ satisfies

$$z^n \overline{P_{n,a}(1/\bar{z})} \equiv P_{n,a}(z) \tag{14}$$

and that all its zeros lies on $|z| = 1$. Indeed, it has a zero of multiplicity $n-2$ at -1 and two simple complex conjugate zeros

$$\left(1\Big/\left(\lambda_a \frac{n^n}{2^2(n-2)^{n-2}} + a\right)\right)\left\{\lambda_a \frac{n^n}{2^2(n-2)^{n-2}} - a \right.$$
$$\left. \pm i\sqrt{4\lambda_a(n^n/2^2(n-2)^{n-2})\, a}\right\}$$

of unit modulus. We claim that $\tau_a < \tau_b$ if $0 < a < b \leqslant 1$. If this was not true the nth degree polynomial $P_{n,b} - P_{n,a}$ would have at least two zeros on $(-1, 1)$ and because of (14) at least two zeros on $\mathbb{R} \backslash [-1, 1]$. Together with a zero of multiplicity $n-2$ at -1 it adds up to at least $n+2$ zeros. But then $P_{n,b}(z) - P_{n,a}(z)$ would be identically zero which is obviously not the case. Counting the zeros of $P_{n,b} - P_{n,a}$ also shows that if $b > a$ then $P_{n,b}(x) > P_{n,a}(x)$ on $[\tau_b, 1]$ and because of (14) also on $[1, 1/\tau_b]$.

Remark 2. It is clear that τ_1 is determined by the system

$$q_{n,n,*}(x) + \lambda_1 q_{n,n-2,*}(x) = 1$$
$$q'_{n,n,*}(x) + \lambda_1 q'_{n,n-2,*}(x) = 0.$$

Eliminating λ_1 we see that τ_1 is a root of the equation

$$\left(1 - \frac{1-x}{2}\right)^n = 1 - \frac{n}{2}\frac{1-x}{2}.$$

In order to estimate $1 - \tau_1$ from below we set $(1-x)/2 = \alpha/n$ which leads us to

$$\left(1 - \frac{\alpha}{n}\right)^n + \frac{\alpha}{2} = 1;$$

i.e.,

$$e^{-\alpha}+\frac{\alpha}{2}-1+\left\{\left(1-\frac{\alpha}{n}\right)^{n}-e^{-\alpha}\right\}=0. \tag{15}$$

At the point α_0 ($=1.59362\ldots$), where $e^{-\alpha}+\alpha/2-1$ vanishes, the left-hand side of (15) is negative and so

$$(1-\tau_1)\,n \geqslant 2\alpha_0 = 3.18724\ldots. \tag{16}$$

1.4. *Analogue of Chebyshev's Inequality*

The following result is to be compared with Chebyshev's inequality (5).

THEOREM 1. *Let $p \in \pi_n$. If $|p(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$, then for $n \geqslant 4$*

$$|p(x)| \leqslant \begin{cases} \max\{-q_{n,n-1,*}(|x|), P_{n,1}(|x|)\} & \text{for} \quad 1<|x|<\dfrac{n}{n-2}, \\ |x|^n & \text{for} \quad |x| \geqslant \dfrac{n}{n-2}, \end{cases} \tag{17}$$

where $P_{n,1}$ is as in (13).

Remark 3. For each $x \in \mathbb{R}\backslash[-1,1]$ we can find a polynomial $p \in \mathscr{P}_n^+$ bounded by 1 on $[-1,1]$ for which the above bound for $|p(x)|$ is attained. This means that (17) cannot be improved even if we restrict ourselves to the subclass $\mathscr{P}_n^+$.

1.5. *Extension of an Observation of Laguerre*

It was observed by Laguerre (see [6]) that if -1, $+1$ are consecutive zeros of a polynomial $p \in \mathscr{P}_n$ with only real zeros then $p'(x) \neq 0$ for $x \in (-1, -1+2/n) \cup (1-2/n, 1)$. Note that $p'(x)$ can vanish at a point x' in $(-1,1)$ only if $|p(x')| = \max_{-1 \leqslant x \leqslant 1} |p(x)|$. The following result, closely related to Theorem 1, constitutes, therefore, a nontrivial extension of Laguerre's observation.

THEOREM 2. *Let $p \in \pi_n$. If $p(1)=0$ and $\max_{-1 \leqslant x \leqslant 1} |p(x)| = M$, then*

$$p(x) \leqslant q_{n,n-1,*}(x)\,p\left(1-\frac{2}{n}\right) < M \qquad \text{for} \quad x \in \left(1-\frac{2}{n}, 1\right). \tag{18}$$

The estimate is sharp and cannot be improved even if we restrict ourselves to polynomials with only real zeros.

Since, according to (18), the maximum of $p(x)$ cannot be attained on $(1-2/n, 1)$, the derivative p' cannot vanish there. It may be observed that this does not depend on the value of p at -1. However, if p had a zero at -1, then by symmetry $p'(x)$ would be different from zero in $(-1, -1+2/n)$ as well.

1.6. *Upper Bound for $p'(x)$ at a Point x in $[-1, 1]$*

In order to state our pointwise estimates for $p'(x)$ we need to introduce some further notations. For $1 \leqslant m \leqslant n$ let

$$\xi_{n,m} := \frac{-n^2 + m(2n-1) - \sqrt{m^2 + 4mn(n-m)}}{n^2}, \tag{19}$$

$$\eta_{n,m} := \frac{-n^2 + n + m(2n-1) - \sqrt{(n-m)^2 + 4mn(n-m)}}{n^2} \tag{20}$$

and

$$t_{n,m}(x) := \frac{nx^2 + (n-2m+1)\,x - 1}{(n-1)\,x + n - 2m + 1} \qquad \left(x \neq -1 + \frac{2(m-1)}{n-1}\right). \tag{21}$$

It is easily checked that $\xi_{n,1} = -1$, $\eta_{n,n} = 1$, $\xi_{n,m} < \eta_{n,m} < \xi_{n,m+1} < \eta_{n,m+1}$ for $1 \leqslant m \leqslant n-1$. Besides, for $0 \leqslant k \leqslant n$, $-1 < t < 1$, let

$$r_{n,k}(t; x) := \frac{q'_{n,k}(x)}{q_{n,k}(t)}. \tag{22}$$

Notice the lack of any obvious symmetry in the location of the points $\xi_{n,m}$, $\eta_{n,m}$.

THEOREM 3. *Let* $\mathscr{E}_n(x) := \sup\{p'(x) : p \in \pi_n,\ p(u) \leqslant 1$ *for* $-1 \leqslant u \leqslant 1\}$. *Then*

$$\mathscr{E}_n(x) = q'_{n,m,*}(x) \qquad \text{if} \quad \xi_{n,m} \leqslant x \leqslant \eta_{n,m}, \qquad 1 \leqslant m \leqslant n, \tag{23}$$

whereas with $t_{n,m}$, $r_{n,m}$ *as above*

$$\mathscr{E}_n(x) = r_{n,m}(t_{n,m}(x); x) \qquad \text{if} \quad \eta_{n,m-1} < x < \xi_{n,m}, \qquad 2 \leqslant m \leqslant n. \tag{24}$$

Further,

$$p'(x) \geqslant -\mathscr{E}_n(-x) \qquad \text{for all} \quad x \in [-1, 1]. \tag{25}$$

The estimates are all sharp. Polynomials for which the bounds are attained have all their zeros on $\mathbb{R} \setminus (-1, 1)$.

Remark 4. Our proof of Theorem 3 will show that for Eq. (23) or (24) to hold we need $p(u) \leqslant 1$ to be satisfied at just one point, depending on x and on n.

1.7. *Extension of* (6) *to Polynomials in* π_n

Although the bound in (12) is asymptotically the best possible it is not sharp for any n. As a simple application of Theorem 3, we shall obtain the following.

COROLLARY 1. *If* $p \in \pi_n$ *and* $|p(x)| \leqslant 1$ *for* $-1 \leqslant x \leqslant 1$, *then* (6) *holds.*

1.8. *Estimates for Some of the Maclaurin Coefficients*

For n, k both odd or both even, satisfying $k(k+1) < n < (k+1)(k+2)$, let

$$Q_{n,k}(x) := (1+x)^{(n+k)/2} \times (1-x)^{(n-k-2)/2} \left(1-(k+1)\frac{(k+1)^2-(n-1)}{2(k+1)^2-n}\,x\right)$$

and

$$Q_{n,k,*}(x) := \frac{(k+1)^{n-1}}{(k+2)^{(n+k)/2}\,k^{(n-k-2)/2}}\,\frac{2(k+1)^2-n}{(k+1)^2-1}\,Q_{n,k}(x).$$

From Theorem 3 we shall deduce the following result.

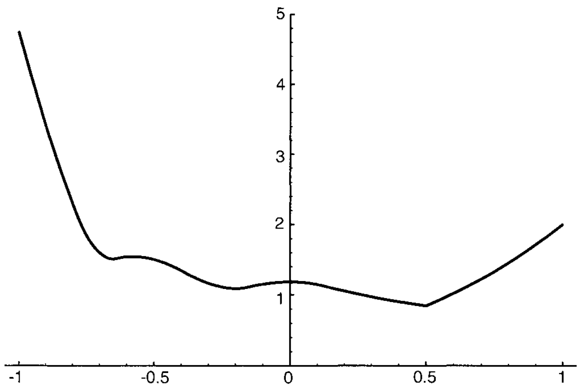

FIG. 1. The function $\mathscr{E}_4(x)$.

TABLE I

$\mathscr{E}_n(0)$ for the First Few Values of n

n	$\mathscr{E}_n(0)$
1	$\frac{1}{2}=0.5$
2	$\frac{1}{2}=0.5$
3	$\frac{8}{9}=0.\bar{8}$
4	$\frac{32}{27}=1.\overline{185}$
5	$\frac{32}{27}=1.\overline{185}$
6	$\frac{729}{512}=1.42382812\ldots$
7	$\frac{2470629}{1600000}=1.54414312\ldots$
8	$\frac{6561}{4096}=1.60180664062\ldots$
9	$\frac{59049}{32768}=1.802032470703\ldots$
10	$\frac{59049}{32768}=1.802032470703\ldots$

COROLLARY 2. *Let* $k(k+1)\leqslant n<(k+1)(k+2)$. *If* $p(x):=\sum_{\nu=0}^{n} a_\nu x^\nu$ *satisfies the conditions of Theorem* 3, *then*

$$|a_1|\leqslant\begin{cases}|q'_{n,\,(n+k)/2,\,*}(0)| & \text{if } n=k(k+1), \text{ where } k \text{ is even,}\\ |q'_{n,\,(n+k+1)/2,\,*}(0)| & \text{if either } n=k(k+1)+2j-1,\\ & \text{where } k \text{ is even,}\\ & 1\leqslant j\leqslant k+1, \text{ or } n=k(k+1)+2j,\\ & \text{where } k \text{ is odd, } 0\leqslant j\leqslant k,\\ |q'_{n-1,\,(n+j-1)/2,\,*}(0)| & \text{if } n \text{ is of the form } j^2+1, \text{ where } j\in\mathbb{N},\\ |Q'_{n,\,k,\,*}(0)| & \text{if } n \text{ does not fall in any}\\ & \text{of the preceding categories.}\end{cases}$$

The estimate cannot be improved even if p satisfies the conditions of Theorem B.

Remark 5. The above bound for $|a_1|$ is the sharp version of (10). It is instructive to compare Corollaries A and 2. In the former, the bound for $|a_1|$ is n if $n-1$ is even and $n-1$ if $n-1$ is odd. In the latter, the dependence of the bound on n is far more complicated.

Although it would be nice to obtain, in the situation of Corollary 2, the sharp upper bound for $|a_\nu|$ for each ν it does not seem to be a simple matter, except when $\nu\in\{0,n\}$.

If $\nu=0$ then $|a_0|=a_0=p(0)\leqslant 1$, the bound being clearly sharp.

If $v=n$ then writing $p(x)$ in the form $\sum_{k=0}^{n} A_k(1+x)^k (1-x)^{n-k}$, where $A_k \geqslant 0$ for $0 \leqslant k \leqslant n$, we see that

$$\begin{aligned}|a_n| &= |A_n - A_{n-1} + A_{n-2} - \cdots| \\ &\leqslant A_n + A_{n-1} + A_{n-2} + \cdots \\ &= p(0) = a_0 \leqslant 1.\end{aligned}$$

Simple examples like $1-x^n$, $(1-x^2)^{n/2}$ show that the bound is attained when n is even. The same can be said when n is odd, except that the example is not so obvious. In Section 2.2.5 we will show that there exists a polynomial

$$\begin{aligned}p_{n,\,(n-1)/2,\,\theta}(x) &:= (1-x)^{(n-1)/2}\,(1+x)^{(n-3)/2} \\ &\quad\times (1+x^2-2x\cos\theta), \qquad 0<\theta<\pi,\end{aligned}$$

belonging to $\mathscr{P}_n^+$ whose maximum on $[-1, 1]$ is attained at the origin. Thus

$$p_{n,\,(n-1)/2,\,\theta}(0) = 1 = \max_{-1\leqslant x\leqslant 1} p_{n,\,(n-1)/2,\,\theta}(x).$$

Since the zeros of $p_{n,\,(n-1)/2,\,\theta}(z) = \sum_{v=0}^{n} a_v z^v$ all lie on $|z|=1$ we must have

$$|a_n| = |a_0| = a_0 = p_{n,\,(n-1)/2,\,\theta}(0) = 1.$$

We also obtain best possible bounds for a_2. Note that a_2 is the same as $\frac{1}{2}p''(0)$.

1.9. *Lower Bound for* $p''(0)$

It is fairly easy to find the exact lower bound for $p''(0)/\max_{-1\leqslant x\leqslant 1} p(x)$ for $p \in \pi_n$. The following result holds.

THEOREM 4. *Let* $p \in \pi_n$. *If* $p(0) \leqslant 1$, *then*

$$p''(0) \geqslant \begin{cases} -n & \text{if } n \text{ is even,} \\ -n+1 & \text{if } n \text{ is odd.} \end{cases} \tag{26}$$

The example $p(x) := (1-x^2)^{[n/2]}$ shows that the estimate is sharp for even as well as for odd n. It cannot be improved even if, instead of $p(0) \leqslant 1$, we assume $|p(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$.

1.10. *Upper Bound for* $p''(0)$

Clearly, any polynomial $p(x) = \sum_{k=0}^{n} A_k(1+x)^k (1-x)^{n-k}$ belonging to π_n can be written as the Riemann–Stieltjes integral

$$p(x) = \int_0^n (1+x)^t\,(1-x)^{n-t}\,d\mu(t), \tag{27}$$

where

$$\mu(t)=\begin{cases}0 & \text{for} \quad t=0\\ A_0 & \text{for} \quad 0<t<1\\ A_0+\cdots+A_{k-1} & \text{for} \quad k-1\leqslant t\leqslant k, \quad 2\leqslant k\leqslant n,\\ A_0+\cdots+A_n & \text{for} \quad t=n.\end{cases}$$

As regards our upper bound for $p''(0)$ we find it convenient to extend the class π_n by allowing μ in (27) to be a step function with nonnegative jumps at an arbitrary set of points

$$0=\mu_0<\mu_1<\cdots<\mu_N=n,$$

where $N\in\mathbb{N}$. By introducing additional jump points with zero jump *we may assume* $\mu_{N-k}=n-\mu_k$ for all k. We denote by $\mathscr{F}_n$ the class π_n so generalized.

In order to present our upper bound for the second derivative at the origin we need to introduce certain parameters and make some preliminary observations.

For $n\geqslant 4$ and $\sqrt{2}\leqslant c\leqslant\sqrt{3}$ let

$$\gamma_n(c):=c\sqrt{n}\log\frac{\sqrt{n}+c}{\sqrt{n}-c}=2c^2\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{c^2}{n}\right)^k,$$

$$\Lambda_n(c):=\sqrt{\frac{\gamma_n(c)}{\gamma_n(c)-4}}.$$

As c increases $\gamma_n(c)$ increases and $\Lambda_n(c)$ decreases. Clearly,

$$\begin{gathered}2c^2\left(1+\frac{1}{3}\frac{c^2}{n}+\frac{1}{5}\frac{c^4}{n^2}\right)<\gamma_n(c)<\frac{2c^2n}{n-c^2},\\ \frac{c^2n}{(c^2-2)\,n+2c^2}<(\Lambda_n(c))^2<\frac{c^2+\frac{1}{3}(c^4/n)+\frac{1}{5}(c^6/n^2)}{c^2-2+\frac{1}{3}(c^4/n)+\frac{1}{5}(c^6/n^2)}.\end{gathered} \tag{28}$$

In particular,

$$(\Lambda_n(\sqrt{3-9/5n}))^2-(\sqrt{3-9/5n})^2<0,$$

$$(\Lambda_n(\sqrt{2}))^2-(\sqrt{2})^2>\frac{n}{2}-2\geqslant 0,$$

and so $\Lambda_n(c)-c$ must have *one and only one* zero in $(\sqrt{2},\sqrt{3-9/5n})$ which we denote by c_n.

THEOREM 5. *Let* $n \geqslant 4$ *be an integer.* (i) *There is one and only one value of* c *(call it* $c_{*,n}$*) in* $(\sqrt{2}, c_n)$ *such that for some positive* α *(say* $\alpha_{*,n}$*) the two equations*

$$\left(\frac{1+c/\sqrt{n}}{1-c/\sqrt{n}}\right)^{2\alpha\sqrt{n}} = \frac{(4\alpha^2-1)\sqrt{n}\log\dfrac{1+c/\sqrt{n}}{1-c/\sqrt{n}}+8\alpha}{(4\alpha^2-1)\sqrt{n}\log\dfrac{1+c/\sqrt{n}}{1-c/\sqrt{n}}-8\alpha}, \tag{29}$$

$$\left(\frac{1+c/\sqrt{n}}{1-c/\sqrt{n}}\right)^{2\alpha\sqrt{n}} = \frac{2\alpha+c}{2\alpha-c} \tag{30}$$

are simultaneously satisfied. (ii) *For each* $f \in \mathscr{F}_n$ *such that* $(f(x)+f(-x))/2 \leqslant 1$ *at* $x=(1/\sqrt{n})\, c_{*,n}$*, we have*

$$f''(0) \leqslant \frac{4\alpha^2-1}{2\alpha}\sqrt{4\alpha^2-c^2}\left(1-\frac{c^2}{n}\right)^{-n/2}\cdot n \qquad (c=c_{*,n},\ \alpha=\alpha_{*,n}). \tag{31}$$

Remark 6. In particular, the above estimate for $f''(0)$ holds if $f(x) \leqslant 1$ at $x = \pm(1/\sqrt{n})\, c_{*,n}$. Our proof will show that (31) cannot be improved even if $f(x) \leqslant 1$ for all $x \in [-1, 1]$. Indeed, there exists a function $f \in \mathscr{F}_n$ bounded by 1 on $[-1, 1]$ for which (31) becomes an equality.

For each $n \geqslant 4$ the constants $c_{*,n}$, $\alpha_{*,n}$ can be calculated numerically with little difficulty but it is desirable to have a bound which does not require any calculations on the part of the reader. We shall prove the following.

THEOREM 5′. *If* c_* $(=1.729228319\ldots)$ *is the unique root of*

$$h(c) := e^{2c^2/\sqrt{c^2-2}} - \frac{1+\sqrt{c^2-2}}{1-\sqrt{c^2-2}} = 0 \tag{32}$$

in $(\sqrt{2}, \sqrt{3})$ *then*

(a) $c_{*,n} < c_*$ *for all* $n \geqslant 4$, *whereas*

(b) $c_* - 1.35/(n-3) < c_{*,n}$ *for all* $n \geqslant 410$.

Further, for each $f \in \mathscr{F}_n$*, where* $n \geqslant 4$*, we have*

$$f''(0) < 0.199631037\ldots\left(1-\frac{c_*^2}{n}\right)^{-n/2}\cdot n \cdot \max_{-1\leqslant x\leqslant 1} f(x). \tag{31′}$$

Remark 7. It is easily seen that the quantity $0.19963\ldots(1-c_*^2/n)^{-n/2}$ appearing in (31′) is less than $0.89032\ldots\exp(9/4(n-3))$ which is itself less than $0.89433\ldots$ if $n \geqslant 500$.

Remark 8. The constant 1.35 and the number 410 appearing in the statement of the preceding theorem have no special significance. Although it is not obvious in itself, it follows trivially from (a) and (b) that

$$c_{*,n} \to c_* \qquad \text{as} \quad n \to \infty.$$

Besides (see (54) below and (28))

$$\alpha_{*,n} \to \alpha_* := \frac{1}{2}\frac{c_*}{\sqrt{c_*^2-2}} \qquad \text{as} \quad n \to \infty.$$

Remark 9. The upper bounds for $f''(0)$ contained in (31) and (31′) are valid for functions belonging to the class $\mathscr{F}_n$ which is larger than π_n and it is reasonable to ask how good they are if f is restricted to π_n. Table II contains numerically calculated lower bounds for

$$\frac{1}{n}\sup_{f \in \pi_n} \frac{f''(0)}{\max_{-1 \leqslant x \leqslant 1} f(x)}$$

for certain values of n. In each case, the polynomial providing the bound is of the form

$$p_{n,k}(x) := \frac{q_{n,k}(x) + q_{n,n-k}(x)}{\max_{-1 \leqslant x \leqslant 1}\{q_{n,k}(x) + q_{n,n-k}(x)\}}.$$

TABLE II

Numerical Estimates for $(1/n)\sup_{f \in \pi_n} f''(0)/\max_{-1 \leqslant x \leqslant 1} f(x)$ Compared with $(1/n)\sup_{f \in \mathscr{F}_n} f''(0)/\max_{-1 \leqslant x \leqslant 1} f(x)$

n	k	$(1/n)\, p''_{n,k}(0)$	$0.19963\ldots(1-c_*^2/n)^{-n/2}$
50	19	0.87582456…	0.93277124…
100	41	0.87749734…	0.91086590…
200	88	0.88499485…	0.90043291…
300	135	0.88809300…	0.89702816…
500	231	0.88705670…	0.89432995…
1000	473	0.88844271…	0.89232103…
5000	2439	0.88998349…	0.89072291…
10000	4913	0.89025089…	0.89052370…
50000	24806	0.89030251…	0.89036443…

2. Proofs of Theorems 1 and 2

To start with we present

2.1. *Proof of Theorem* 2

Since $p \in \pi_n$ and $p(1)=0$ we have $p(x) := \sum_{k=0}^{n-1} A_k q_{n,k}(x)$, where $A_k \geqslant 0$ for $k=0, 1, ..., n-1$. Now note that if $1-2/n < x < 1$, then for $k=0, 1, ..., n-1$,

$$\frac{q_{n,k}(x)}{q_{n,k}(1-2/n)} = \left(\frac{n}{2}\right)^n \frac{(1+x)^k\,(1-x)^{n-k}}{(n-1)^k}$$
$$\leqslant \left(\frac{n}{2}\right)^n \frac{(1+x)^{n-1}\,(1-x)}{(n-1)^{n-1}} = q_{n,\,n-1,\,*}(x)$$

and so

$$p(x) = \sum_{k=0}^{n-1} A_k \frac{q_{n,k}(x)}{q_{n,k}(1-2/n)}\, q_{n,k}\left(1-\frac{2}{n}\right)$$
$$\leqslant q_{n,\,n-1,\,*}(x) \sum_{k=0}^{n-1} A_k q_{n,k}\left(1-\frac{2}{n}\right)$$
$$= q_{n,\,n-1,\,*}(x)\, p\left(1-\frac{2}{n}\right).$$

At each point of $(1-2/n, 1)$ equality holds for $Mq_{n,\,n-1,\,*}$.

Remark 10. The preceding argument also shows that if $p \in \pi_n$ and $p(1)=0$, then $|p(z)| \leqslant |q_{n,\,n-1,\,*}(z)|\; p(1-2/n)$ for $|z-1-2/n(n-2)| \leqslant 2(n-1)/n(n-2)$.

2.2. *Proof of Theorem* 1

It is clearly enough to prove (17) for $1 < x < \infty$.

2.2.1. *Lower Bound for* $p(x)$ *on* $(1, n/(n-2))$. Let $p(1)=a$. For $x \in (1, n/(n-2))$ we have

$$p(x) = a\left(\frac{1+x}{2}\right)^n + \sum_{k=0}^{n-1} A_k (1+x)^k\,(1-x)^{n-k}$$
$$\geqslant a\left(\frac{1+x}{2}\right)^n + \sum_{0 \leqslant k \leqslant n-1,\; n-k \text{ odd}} A_k (1+x)^k\,(1-x)^{n-k}$$
$$= a\left(\frac{1+x}{2}\right)^n - x^n \sum_{0 \leqslant k \leqslant n-1,\; n-k \text{ odd}} A_k \left(1+\frac{1}{x}\right)^k \left(1-\frac{1}{x}\right)^{n-k}$$
$$\geqslant a\left(\frac{1+x}{2}\right)^n - x^n \sum_{k=0}^{n-1} A_k (1+t)^k\,(1-t)^{n-k},$$

where $t := 1/x \in ((n-2)/n, 1)$. Note that

$$\sum_{k=0}^{n-1} A_k(1+\cdot)^k (1-\cdot)^{n-k} = p(\cdot) - a\left(\frac{1+\cdot}{n}\right)^n$$

belongs to π_n and vanishes at the point 1. Applying Theorem 2 we conclude that for $(n-2)/n < t < 1$,

$$\sum_{k=0}^{n-1} A_k(1+t)^k (1-t)^{n-k} \leqslant q_{n,n-1,*}(t) \left\{ p\left(1-\frac{2}{n}\right) - a\left(1-\frac{1}{n}\right)^n \right\}.$$

Consequently, if $x \in (1, n/(n-2))$ then

$$\begin{aligned} p(x) &\geqslant a\left(\frac{1+x}{2}\right)^n - x^n q_{n,n-1,*}\left(\frac{1}{x}\right)\left\{p\left(1-\frac{2}{n}\right) - a\left(1-\frac{1}{n}\right)^n\right\} \\ &= q_{n,n-1,*}(x)\, p\left(1-\frac{2}{n}\right) + a\left(\frac{1+x}{2}\right)^n - a\left(1-\frac{1}{n}\right)^n q_{n,n-1,*}(x) \qquad (33) \\ &\geqslant q_{n,n-1,*}(x)\, p(1-2/n) \geqslant q_{n,n-1,*}(x). \qquad (34) \end{aligned}$$

2.2.2. *Upper Bound for* $p(x)$ *on* $(1, 1/\tau_1)$ *as a Function of* $p(1)$. The number τ_a and the polynomial $P_{n,a}$ appearing in this section and the next have been defined for $a \in (0, 1]$ in 1.3.1. Assuming that $p(1) = a$, we obtain the sharp upper bound for $p(x)$ on $(1, 1/\tau_a]$. For $x \in (1, 1/\tau_a]$ we clearly have

$$\begin{aligned} p(x) &= a\left(\frac{1+x}{2}\right)^n + \sum_{l=1}^{n} A_{n-l}(1+x)^{n-l}(1-x)^l \\ &\leqslant a\left(\frac{1+x}{2}\right)^n + \sum_{j=1}^{[n/2]} A_{n-2j}(1+x)^{n-2j}(1-x)^{2j} \\ &= a\left(\frac{1+x}{2}\right)^n + x^n \sum_{j=1}^{[n/2]} A_{n-2j}(1+t)^{n-2j}(1-t)^{2j}, \end{aligned}$$

where $t := 1/x \in [\tau_a, 1)$. Now we have to determine how large (the expression) $\varphi(t) := \sum_{j=1}^{[n/2]} A_{n-2j}(1+t)^{n-2j}(1-t)^{2j}$ can be at a point $t \in [\tau_a, 1)$ if it does not exceed $1 - a((1+\tau_a)/2)^n$ at the point τ_a. The answer is

$$\begin{aligned} \varphi(t) &\leqslant \max_{1 \leqslant j \leqslant [n/2]} \frac{(1+t)^{n-2j}(1-t)^{2j}}{(1+\tau_a)^{n-2j}(1-\tau_a)^{2j}} \sum_{j=1}^{[n/2]} A_{n-2j}(1+\tau_a)^{n-2j}(1-\tau_a)^{2j} \\ &= \frac{(1+t)^{n-2}(1-t)^2}{(1+\tau_a)^{n-2}(1-\tau_a)^2} \sum_{j=1}^{[n/2]} A_{n-2j}(1+\tau_a)^{n-2j}(1-\tau_a)^{2j} \\ &\leqslant \frac{(1+t)^{n-2}(1-t)^2}{(1+\tau_a)^{n-2}(1-\tau_a)^2} \left\{1 - a\left(\frac{1+\tau_a}{2}\right)^n\right\}. \end{aligned}$$

Recalling that

$$a\left(\frac{1+\tau_a}{2}\right)^n+\lambda_a\frac{n^n}{2^n 2^2(n-2)^{n-2}}(1+\tau_a)^{n-2}(1-\tau_a)^2=1,$$

we obtain

$$\varphi(t)\leqslant\lambda_a\frac{n^n}{2^n 2^2(n-2)^{n-2}}(1+t)^{n-2}(1-t)^2$$

and

$$\begin{aligned}p(x)&\leqslant a\left(\frac{1+x}{2}\right)^n+x^n\lambda_a\frac{n^n}{2^n 2^2(n-2)^{n-2}}(1+t)^{n-2}(1-t)^2\\&=a\left(\frac{1+x}{2}\right)^n+\lambda_a\frac{n^n}{2^2(n-2)^{n-2}}\left(\frac{1+x}{2}\right)^{n-2}\left(\frac{1-x}{2}\right)^2\\&=P_{n,a}(x).\end{aligned}$$

2.2.3. *Inequality* (17) *for* $1<x<n/(n-2)$. In Section 1.3.1 it was shown that $\tau_a\leqslant\tau_1<1/\tau_1\leqslant 1/\tau_a$ and that $P_{n,a}(x)\leqslant P_{n,1}(x)$ for $\tau_1\leqslant x\leqslant 1/\tau_1$. Hence

$$p(x)\leqslant P_{n,1}(x)\qquad\text{for}\quad 1<x\leqslant 1/\tau_1. \tag{35}$$

In view of (16), the lower bound in (34) and the upper bound in (35) imply that

$$|p(x)|\leqslant\max\{-q_{n,\,n-1,\,*}(x),\,P_{n,1}(x)\}\qquad\text{for}\quad 1<x<n/(n-2). \tag{36}$$

It is easily seen that the equation

$$q_{n,\,n-1,\,*}(x)+P_{n,\,1}(x)=0$$

has a root u in $(1,n/(n-2))$ and that the bound in (36) is attained on $(1,u]$ by $P_{n,1}$, whereas $q_{n,\,n-1,\,*}$ is extremal on $[u,n/(n-2))$. On $[-u,-1)$ the bound is attained by the polynomial $P_{n,\,1}(-x)$ and on $(-n/(n-2),-u]$ by $q_{n,\,1,\,*}$.

2.2.4. *Inequality* (17) *for* $|x|\geqslant n/(n-2)$. It is much easier to obtain the *sharp* estimate for $|p(x)|$ when $|x|\geqslant n/(n-2)$. Indeed, for all $x\notin[-1,1]$ we have

$$\begin{aligned}|p(x)|&\leqslant\sum_{k=0}^{n}A_k\,|1+x|^k\,|1-x|^{n-k}\\&=|x|^n\sum_{k=0}^{n}A_k\left(1+\frac{1}{x}\right)^k\left(1-\frac{1}{x}\right)^{n-k}\\&=|x|^n\,p(1/x)\leqslant|x|^n.\end{aligned}$$

2.2.5. *Sharpness of* (17) *for* $|x| \geqslant n/(n-2)$. It remains to show that at each point $x \in \mathbb{R} \backslash (-n/(n-2), n/(n-2))$ the bound $|x|^n$ is attained for some $p \in \mathscr{P}_n^+$. Since polynomials in $\mathscr{P}_n^+$ which have all their zeros on $|z|=1$ satisfy $z^n \overline{p(1/\bar{z})} \equiv \pm p(z)$ it is enough to show that given any point $x_0 \in [-1+2/n, 1-2/n]$ there exists a polynomial $p \in \mathscr{P}_n^+$ having all its zeros on $|z|=1$ such that $p(x_0) = \max_{-1 \leqslant x \leqslant 1} p(x)$. Besides, due to obvious symmetry we only need to consider $x_0 \in [0, 1-2/n]$.

First let $x_0 = 1 - 2j/n$, where $1 \leqslant j \leqslant n-1$. The polynomial $q_{n, n-j}$ has the desired property.

Now let $x_0 \in (1 - 2(j+1)/n, 1 - 2j/n)$ for some $j \in \{1, ..., n-3\}$ and consider the family of polynomials

$$p_{n, j, \theta}(x) := (1-x)^j (1+x)^{n-j-2} (1+x^2-2x \cos \theta) \qquad (0 \leqslant \theta \leqslant \pi).$$

The derivative $p'_{n, j, \theta}$ vanishes at a point in $(-1, 1)$ if and only if

$$f_{n, j, \theta}(x) := -nx^3 + (n-2j-2+2(n-1) \cos \theta)\, x^2 + (-n+4-2(n-2j-2) \cos \theta)\, x + n - 2j - 2 - 2 \cos \theta = 0.$$

It is well known (see, for example, [5, Section 43]) that the cubic

$$ax^3 + bx^2 + cx + d$$

has one and only one real zero if and only if

$$\Delta := -4ac^3 + b^2c^2 - 4b^3d + 18abcd - 27a^2d^2 < 0.$$

Writing

$$n - 2j - 2 = (n-4)\, u, \qquad \cos \theta = t,$$

where u and t belong to $[-1, 1]$, the cubic $f_{n, j, \theta}$ takes the form

$$q_{n, u, t}(x) := -nx^3 + ((n-4)\, u + 2(n-1)\, t)\, x^2 - (n-4)(1+2ut)\, x + (n-4)\, u - 2t.$$

Its discriminant $\Delta = \Delta(t)$ equals $4(t^2-1)\, \delta(t)$, where

$$\begin{aligned} \delta(t) := {} & 4(n-1)^2 \{4(n-1) + (n-4)^2 u^2\}\, t^2 \\ & - 4(n-4)\, u\{(n+2)^2 (n-1) + (n+1)(n-4)^2 u^2\}\, t \\ & + (n-4)^2 \{n(n-4) + 2(n^2+10n-2)\, u^2 + (n-4)^2 u^4\}. \end{aligned}$$

Hence for $t \in (-1, 1)$ the sign of $\Delta(t)$ is opposite to that of $\delta(t)$. The discriminant of the quadratic $\delta(t)$ is

$$64n(n-4)^3 \{(n-4)\, u^2 - (n-1)\}^3$$

which is negative if $n > 4$ and then the sign of $\delta(t)$ is the same as that of

$$n(n-4) + 2(n^2 + 10n - 2)\, u^2 + (n-4)^2\, u^4$$

which is positive if $n > 4$. Hence $\Delta(t) < 0$ for $t \in (-1, 1)$ if $n > 4$. If $n = 4$ then $\Delta(t) = 1728(t^2 - 1)\, t^2 < 0$ for $t \in (-1, 1)$. Thus $\Delta(t) < 0$ on $(-1, 1)$ for all admissible values of n.

It follows that for $\theta \in [0, \pi]$ the derivative $p'_{n,j,\theta}$ has one and only one real zero x'_θ other than ± 1 which, by the Gauss–Lucas theorem must lie on $(-1, 1)$. Since x'_θ is a continuous function of θ, taking the value $-1 + 2(j+1)/n$ at 0 and $-1 + 2j/n$ at π it takes every *intermediate value* for some θ in $(0, \pi)$. In other words, every value x_0 in $(1 - 2(j+1)/n, 1 - 2j/n)$ is a zero of $p'_{n,j,\theta}$ for some θ, say θ_0 in $(0, \pi)$. Clearly, then $p_{n,j,\theta_0}(x_0) = \max_{-1 \leqslant x \leqslant 1} p_{n,j,\theta_0}(x)$. The points $1 - 2j/n$, $1 \leqslant j \leqslant n-1$, and the intervals $(1 - 2(j+1)/n, 1 - 2j/n)$, $1 \leqslant j \leqslant n-3$, together cover $[0, 1 - 2/n]$ completely.

3. Proof of Theorem 3

3.1. *The Upper Bound for $p'(x)$*

3.1.1. *The Case $m = n$ of* (23). If $\xi_{n,n} = 1 - 2/n \leqslant x \leqslant 1$ then $q'_{n,k}(x) \leqslant 0$ for $0 \leqslant k \leqslant n-1$ and so

$$p'(x) = \sum_{k=0}^{n} A_k q'_{n,k}(x) \leqslant A_n q'_{n,n}(x) = \frac{p(1)}{q_{n,n}(1)}\, q'_{n,n}(x) = p(1)\, q'_{n,n,*}(x).$$

Since equality holds in the above inequality for the polynomial $q_{n,n,*}$, it follows that $\mathscr{E}_n(x) = q'_{n,n,*}(x)$ if $x \in [\xi_{n,n}, \eta_{n,n}]$. Note that $p'(x) \leqslant q'_{n,n,*}(x)$ for $\xi_{n,n} \leqslant x \leqslant 1$ even if we *only* assume $p(1) \leqslant 1$.

3.1.2. *Proof of* (23) *for* $1 \leqslant m \leqslant n-1$.

Step 1. For each t in $(-1, 1)$ we can write

$$p'(x) = \sum_{k=0}^{n} r_{n,k}(t; x) \cdot A_k q_{n,k}(t),$$

where $r_{n,k}$ is as in (22). Hence

$$p'(x) \leqslant \inf_{-1<t<1} \max_{0 \leqslant k \leqslant n} r_{n,k}(t;x) \cdot p(t). \tag{37}$$

In order to obtain the desired upper bound it is therefore sufficient to show that if $\xi_{n,m} \leqslant x \leqslant \eta_{n,m}$, $1 \leqslant m \leqslant n-1$, then

$$\max_{0 \leqslant k \leqslant n} r_{n,k}\left(-1+\frac{2m}{n};x\right) = r_{n,m}\left(-1+\frac{2m}{n};x\right) = q'_{n,m,*}(x). \tag{38}$$

Remark 11. Since $r_{n,0}(t;x) := -n((1-x)^{n-1}/(1-t)^n) \leqslant 0$ for all $x \in [-1,1]$, whereas $r_{n,n}(t;x) := n((1+x)^{n-1}/(1+t)^n) \geqslant 0$ for all $x \in [-1,1]$ the maximum of $r_{n,k}(-1+2m/n;x)$ for $k \in \{0, 1, ..., n\}$ is the same as $\max_{1 \leqslant k \leqslant n} r_{n,k}(-1+2m/n;x)$.

Step 2. By definition,

$$r_{n,k}(t;x) := \frac{(1+x)^{k-1}(1-x)^{n-k-1}(2k-n-nx)}{(1+t)^k(1-t)^{n-k}} \qquad \text{for} \quad 1 \leqslant k \leqslant n-1.$$

Hence for $2 \leqslant k \leqslant n-1$,

$$r_{n,k}(t;x) - r_{n,k-1}(t;x) = \frac{2(1+x)^{k-2}(1-x)^{n-k-1}}{(1+t)^k(1-t)^{n-k+1}} R_{n,k}(t;x), \tag{39}$$

where

$$R_{n,k}(t;x) := (n-2k+1)t + 1 - (n-2k+1-(n-1)t)x - nx^2. \tag{40}$$

In particular, for $1 \leqslant m \leqslant n-1$ we have

$$R_{n,k}\left(-1+\frac{2m}{n};x\right) = \frac{1}{n}\{(n-2k)(2m-n)+2m + (2m(n-1)-2n(n-k))x - n^2x^2\}.$$

It is easily seen that

$$R_{n,k}\left(-1+\frac{2m}{n};-1\right) = -\frac{4m}{n}(k-1) < 0,$$

whereas

$$R_{n,k}\left(-1+\frac{2m}{n};-1+\frac{2(k-1)}{n}\right) = \frac{4}{n^2}(n-m)(k-1) > 0.$$

Further, $R_{n,k}(-1+2m/n; x) \to -\infty$ as $x \to \infty$ and, therefore, $R_{n,k}(-1+2m/n; \cdot)$ has one and only one zero,

$$x_{n,k,m} := \frac{m(n-1)-n(n-k)-\sqrt{n^2(m-k)^2+m^2+2mn(2n-k-m)}}{n^2},$$

in $(-1, -1+2(k-1)/n)$. Thus $R_{n,k}(-1+2m/n; x)$ is negative for $x < x_{n,k,m}$ and positive at least for $x_{n,k,m} < x \leqslant -1+2(k-1)/n$. It is also positive for $-1+2(k-1)/n < x \leqslant -1+2k/n$ since $r_{n,k}(-1+2m/n; x) \geqslant 0$ for $x \leqslant -1+2k/n$, whereas $r_{n,k-1}(-1+2m/n; x) < 0$ for $x > -1+2(k-1)/n$.

The definition of $x_{n,k,m}$ as the zero of $r_{n,k}(-1+2m/n; \cdot) - r_{n,k-1}(-1+2m/n; \cdot)$ in $(-1, -1+2(k-1)/n)$ extends to $k=n$, including $n=2$. Indeed, a similar calculation shows that $r_{n,n}(-1+2m/n; x) - r_{n,n-1}(-1+2m/n; x)$ is negative for $-1 < x < -1+2m(n-1)/n^2 =: x_{n,n,m}$ and positive for $x_{n,n,m} < x < 1$.

Next we note that if we set $x_{n,1,m} := -1$, $x_{n,n+1,m} := 1$, then

$$x_{n,k,m} < x_{n,k+1,m} \qquad \text{for} \quad 1 \leqslant k \leqslant n. \tag{41}$$

This is clearly true for $k \in \{1, n\}$. Besides, it is easily checked that

$$\begin{aligned} x_{n,n-1,m} &= \frac{m(n-1)-n-\sqrt{n^2(n-m-1)^2+m^2+2mn(n-m+1)}}{n^2} \\ &< -1+\frac{2m(n-1)}{n^2} = x_{n,n,m}. \end{aligned}$$

If $2 \leqslant k \leqslant n-2$, then

$$\begin{aligned} &R_{n,k+1}\left(-1+\frac{2m}{n}; x_{n,k,m}\right) \\ &\quad = R_{n,k+1}\left(-1+\frac{2m}{n}; x_{n,k,m}\right) - R_{n,k}\left(-1+\frac{2m}{n}; x_{n,k,m}\right) \\ &\quad = \frac{1}{n}\left\{2(2m-n)+2nx_{n,k,m}\right\} \\ &\quad = \frac{2}{n^2}\left\{n(k-m)-m-\sqrt{(m-k)^2\, n^2+m^2+2mn(2n-k-m)}\right\} < 0. \end{aligned}$$

As such, $x_{n,k,m}$ cannot belong to $[x_{n,k+1,m}, \; -1+2(k+1)/n]$. Since $x_{n,k,m} < -1+2(k+1)/n$ we must have $x_{n,k,m} < x_{n,k+1,m}$.

Step 3. Now we claim that if $x_{n,k,m} \leqslant x \leqslant x_{n,k+1,m}$, where $1 \leqslant k \leqslant n$, then

$$r_{n,j}\left(-1+\frac{2m}{n}; x\right) \leqslant r_{n,k}\left(-1+\frac{2m}{n}; x\right) \qquad \text{for all } j. \tag{42}$$

First, let $j > k$. Then for $j \geqslant l > k$ the inequality

$$r_{n,l-1}\left(-1+\frac{2m}{n}; x\right) \geqslant r_{n,l}\left(-1+\frac{2m}{n}; x\right)$$

holds for $x \leqslant x_{n,l,m}$ and so certainly for $x \leqslant x_{n,k+1,m}$. Now suppose, if possible, that (42) fails for some $x' \in [x_{n,k,m}, x_{n,k+1,m}]$ and some $j < k$. Let j^* be the *largest* such j. Since $r_{n,j^*}(-1+2m/n; x) < 0$ for $x \in (-1+2j^*/n, 1]$, whereas $r_{n,k}(-1+2m/n; x) \geqslant 0$ for x in $(-1, -1+2k/n]$ and so for $x \in [x_{n,k,m}, x_{n,k+1,m}]$, the point x' must lie in

$$[x_{n,k,m}, x_{n,k+1,m}] \cap [-1, -1+2j^*/n]$$

which (if not empty) is contained in $[x_{n,k,m}, -1+2j^*/n]$; but then $r_{n,j^*+1}(-1+2m/n; x') \geqslant r_{n,j^*}(-1+2m/n; x') > r_{n,k}(-1+2m/n; x')$. This is a contradiction since j^* was supposed to be the largest integer $(<k)$ for which (42) does not hold for all x in $[x_{n,k,m}, x_{n,k+1,m}]$.

In particular, if $x_{n,m,m} = \xi_{n,m} \leqslant x \leqslant \eta_{n,m} = x_{n,m+1,m}$ then

$$r_{n,k}\left(-1+\frac{2m}{n}; x\right) \leqslant r_{n,m}\left(-1+\frac{2m}{n}; x\right) \qquad \text{for all } k;$$

i.e., (38) holds. From (37) and (38) it really follows that if $p \in \pi_n$ and $1 \leqslant m \leqslant n-1$, then for $\xi_{n,m} \leqslant x \leqslant \eta_{n,m}$ we have

$$p'(x) \leqslant q'_{n,m,*}(x) \cdot p\left(-1+\frac{2m}{n}\right) \leqslant q'_{n,m,*}(x) \max_{-1 \leqslant x \leqslant 1} p(x). \tag{23'}$$

Equality holds in both the equalities in (23′) when p is a positive multiple of $q_{n,m}$.

According to the first inequality in (23′), if $p \in \pi_n$ then the sharp inequality "$p'(x) \leqslant q'_{n,m,*}(x)$ for $\xi_{n,m} \leqslant x \leqslant \eta_{n,m}$, $1 \leqslant m \leqslant n-1$" contained in (23) holds even if we *only* assume $p(-1+2m/n) \leqslant 1$.

3.1.3. *Proof of* (24). In view of (37) it suffices to show that if $\eta_{n,m-1} < x < \xi_{n,m}$, $2 \leqslant m \leqslant n$, then

$$\inf_{-1<t<1} \max_{0 \leqslant j \leqslant n} r_{n,j}(t; x) = r_{n,m}(t_{n,m}(x); x). \tag{43}$$

Step 1. Let $x \in (-1, 1-2/n)$ be arbitrary but fixed. If k is the smallest integer such that $k-1 > n(1+x)/2$, or equivalently $-1+(2k-2)/n \leqslant x < -1+2(k-1)/n$, then $r_{n,j}(t; x) \leqslant 0$ for $0 \leqslant j \leqslant k-2$ and so

$$M(t; x) := \max_{0 \leqslant j \leqslant n} r_{n,j}(t; x) = \max_{k-1 \leqslant j \leqslant n} r_{n,j}(t; x).$$

Since $x \neq \pm 1$, formula (39), where $R_{n,k}$ is defined in (40), is valid for all $k \in \{1, ..., n\}$. We note that for $j-1 > n(1+x)/2$,

$$\lim_{t \downarrow -1} R_{n,j}(t; x) = (2j-n-nx)(1+x) > 0,$$

whereas

$$\lim_{t \uparrow +1} R_{n,j}(t; x) = \{n(1+x) - 2(j-1)\}(1-x) < 0.$$

Since $R_{n,j}(t; x)$ is linear in t it follows that there exists one and only one number $t_{n,j}(x)$ such that

$$R_{n,j}(t_{n,j}(x); x) = 0 \qquad (j-1 > n(1+x)/2). \tag{44}$$

We set $t_{n,k-1}(x) := 1$, $t_{n,n+1}(x) := -1$. Then $t_{n,j}(x)$ is defined for $k-1 \leqslant j \leqslant n+1$ and we have

$$t_{n,n+1}(x) := -1 < t_{n,n}(x) < t_{n,n-1}(x) < \cdots < t_{n,k}(x) < 1 =: t_{n,k-1}(x). \tag{45}$$

To see this, note that for $k \leqslant j \leqslant n$, we have

$$R_{n,j}(x; x) = 1 - x^2 > 0$$

and, therefore, $x < t_{n,j}(x)$. It follows that if $k \leqslant j < n$, then

$$\begin{aligned} R_{n,j+1}(t_{n,j}(x); x) &= R_{n,j+1}(t_{n,j}(x); x) - R_{n,j}(t_{n,j}(x); x) \\ &= -2(t_{n,j}(x) - x) \end{aligned}$$

is negative. Hence, $t_{n,j+1}(x) < t_{n,j}(x)$ for $k-1 \leqslant j \leqslant n$.

Step 2. As seen from (45), the points $t_{n,n+1}(x), t_{n,n}(x), ..., t_{n,k}(x), t_{n,k-1}(x)$ constitute a partition of the interval $[-1, 1]$ which defines $n-k+2$ subintervals $[t_{n,j+1}(x), t_{n,j}(x)]$, $k-1 \leqslant j \leqslant n$. We claim that if $t \in [t_{n,j+1}(x), t_{n,j}(x)]$, then

$$\max_{v \geqslant k-1} r_{n,v}(t; x) = r_{n,j}(t; x). \tag{46}$$

Indeed, by the definition of $t_{n,\cdot}(x)$ as the root of $R_{n,\cdot}(t;x)=0$, we have

$$r_{n,j}(t;x) \geqslant r_{n,j-1}(t;x) \qquad \text{for} \quad -1 \leqslant t \leqslant t_{n,j}(x),$$
$$r_{n,j-1}(t;x) \geqslant r_{n,j-2}(t;x) \qquad \text{for} \quad -1 \leqslant t \leqslant t_{n,j-1}(x),$$
$$\text{and so for } -1 \leqslant t \leqslant t_{n,j}(x), \text{ etc.}$$

Further,

$$r_{n,j}(t;x) \geqslant r_{n,j+1}(t;x) \qquad \text{for} \quad t_{n,j+1}(x) \leqslant t \leqslant 1,$$
$$r_{n,j+1}(t;x) \geqslant r_{n,j+2}(t;x) \qquad \text{for} \quad t_{n,j+2}(x) \leqslant t \leqslant 1,$$
$$\text{and so for } t_{n,j+1}(x) \leqslant t \leqslant 1, \text{ etc.}$$

Hence, (46) holds for $t \in [t_{n,j+1}(x), t_{n,j}(x)]$.

Step 3. We also need to note that

$$-1+\frac{2(m-1)}{n} < t_{n,m}(x) < -1+\frac{2m}{n} \qquad \text{if} \quad x \in (\eta_{n,m-1}, \xi_{n,m}), \quad 2 \leqslant m \leqslant n. \tag{47}$$

Recall that $\xi_{n,m}=x_{n,m,m}$, $\eta_{n,m-1}=x_{n,m,m-1}$. By the definition of $x_{n,m,m}$ we have

$$R_{n,m}\left(-1+\frac{2m}{n};x\right)<0 \qquad \text{if} \quad x \in (-1, x_{n,m,m}),$$

and so

$$t_{n,m}(x) < -1+2m/n.$$

Besides, if $x \in (x_{n,m,m-1}, -1+2m/n]$ then $R_{n,m}(-1+2(m-1)/n;x)>0$, which implies that

$$-1+\frac{2(m-1)}{n} < t_{n,m}(x) \qquad \text{for} \quad x_{n,m,m-1} < x < x_{n,m,m} < -1+2m/n.$$

Step 4. Now we are ready to prove (43). Assume $x \in (x_{n,m,m-1}, x_{n,m,m})$ to be arbitrary but fixed and recall that

$$M(t;x) := \max_{0 \leqslant j \leqslant n} r_{n,j}(t;x) = \max_{k-1 \leqslant j \leqslant n} r_{n,j}(t;x),$$

where k is the smallest integer such that $k-1 > n(1+x)/2$. By definition,

$$M(t;x) \geqslant r_{n,m}(t;x) = \frac{q'_{n,m}(x)}{q_{n,m}(t)} \qquad (-1<t<1).$$

The function $q_{n,m}(t)$ is increasing on $(-1, -1+2m/n]$ and so (see (45), (47)) on $(-1, t_{n,m}(x)]$. Hence $q'_{n,m}(x)/q_{n,m}(t)$ *decreases* to $q'_{n,m}(x)/q_{n,m}(t_{n,m}(x))$ $(=r_{n,m}(t_{n,m}(x); x))$ as t increases to $t_{n,m}(x)$. In particular,

$$M(t; x) \geqslant r_{n,m}(t_{n,m}(x); x) \qquad \text{for} \quad t \in (-1, t_{n,m}(x)], \tag{48}$$

where (see (46)) equality holds only for $t = t_{n,m}(x)$. Again by definition,

$$M(t; x) \geqslant r_{n,m-1}(t; x) = \frac{q'_{n,m-1}(x)}{q_{n,m-1}(t)} \qquad (-1 < t < 1).$$

The function $q_{n,m-1}(t)$ is decreasing on $[-1+2(m-1)/n, 1)$ and so (see (47)) on $[t_{n,m}(x), 1)$. This implies that $q'_{n,m-1}(x)/q_{n,m-1}(t)$ is an increasing function of t on $[t_{n,m}(x), 1)$. In particular,

$$M(t; x) \geqslant r_{n,m-1}(t_{n,m}(x); x) \qquad \text{for} \quad t \in [t_{n,m}(x), 1),$$

where (see (46)) equality holds only for $t = t_{n,m}(x)$. By definition, $t_{n,m}(x)$ is the root of $R_{n,m}(t; x) = 0$; i.e., $r_{n,m-1}(t_{n,m}(x); x) = r_{n,m}(t_{n,m}(x); x)$. Hence,

$$M(t; x) \geqslant r_{n,m}(t_{n,m}(x); x) \qquad \text{for} \quad t \in [t_{n,m}(x), 1), \tag{49}$$

Inequalities (48) and (49) together imply (43).

From (37) and (43) it really follows that if $p \in \pi_n$ and $2 \leqslant m \leqslant n$, then for $\eta_{n,m-1} < x < \xi_{n,m}$ we have

$$p'(x) \leqslant r_{n,m}(t_{n,m}(x); x) \cdot p(t_{n,m}(x)) \leqslant r_{n,m}(t_{n,m}(x); x) \cdot \max_{-1 \leqslant x \leqslant 1} p(x). \tag{24'}$$

Equality can hold in both the inequalities for the same polynomial in π_n. Indeed, we show that for each given x_0 in $(\eta_{n,m-1}, \xi_{n,m})$ there exists a polynomial p of degree at most n whose zeros are all real, $p(x) > 0$ for $-1 < x < 1$, $\max_{-1 \leqslant x \leqslant 1} p(x) = 1 = p(t_{n,m}(x_0))$, and

$$p'(x_0) = r_{n,m}(t_{n,m}(x_0); x_0). \tag{50}$$

Step 5. For x in $(\eta_{n,m-1}, \xi_{n,m})$, let

$$\theta(x) := -\frac{(n-1)\,t + n - 2m + 1}{nt^2 + (n-2m+1)\,t - 1} \qquad (t = t_{n,m}(x)).$$

Not only does (47) hold, but it can be easily seen that as x increases from $\eta_{n,m-1}$ to $\xi_{n,m}$ the quantity $t_{n,m}(x)$ increases from $-1+2(m-1)/n$ to $-1+2m/n$ and $\theta(x)$ increases from -1 to $+1$. Note that $\theta(x) = 0$ if and only if $t = -1 + 2(m-1)/(n-1)$.

Given any x_0 in $(\eta_{n,m-1}, \xi_{n,m})$ let

$$q_{n,m,\theta_0}(x) := (1+x)^{m-1}(1-x)^{n-m}(1+\theta_0 x) \qquad (\theta_0 = \theta(x_0))$$

and

$$q_{n,m,\theta_0,*} := \frac{q_{n,m,\theta_0}(x)}{q_{n,m,\theta_0}(t_0)} \qquad (t_0 = t_{n,m}(x_0)). \tag{51}$$

It is a matter of simple calculation that

$$\begin{aligned} q'_{n,m,\theta_0}(x) = &-(1+x)^{m-2}(1-x)^{n-m-1} \\ &\times \{n\theta_0 x^2 + (n-1+(n-2m+1)\,\theta_0)\,x + n-2m+1-\theta_0\} \end{aligned}$$

and so

$$\begin{aligned} q'_{n,m,\theta_0}(t_0) = &-(1+t_0)^{m-2}(1-t_0)^{n-m-1} \\ &\times \{(nt_0^2 + (n-2m+1)\,t_0 - 1)\,\theta_0 \\ &+ (n-1)\,t_0 + n-2m+1\} = 0 \end{aligned}$$

which implies that $q_{n,m,\theta_0,*}(x)$ assumes its maximum on $[-1, 1]$ at $x = t_0$. Further, we observe that

$$\begin{aligned} &r_{n,m}(t_0; x_0) - q'_{n,m,\theta_0,*}(x_0) \\ &\quad = \frac{(1+x_0)^{m-1}(1-x_0)^{n-m-1}(2m-n-nx_0)}{(1+t_0)^m(1-t_0)^{n-m}} \\ &\qquad + \frac{(1+x_0)^{m-2}(1-x_0)^{n-m-1}}{(1+t_0)^{m-1}(1-t_0)^{n-m}(1+\theta_0 t_0)} \\ &\qquad \times \{n\theta_0 x_0^2 + (n-1+(n-2m+1)\,\theta_0)\,x_0 + n-2m+1-\theta_0\} \\ &\quad = (1-\theta_0)\frac{(1+x_0)^{m-2}(1-x_0)^{n-m-1}}{(1+t_0)^m(1-t_0)^{n-m}(1+\theta_0 t_0)} R_{n,m}(t_0; x_0) = 0; \end{aligned}$$

i.e., $q'_{n,m,\theta_0,*}(x_0) = r_{n,m}(t_{n,m}(x_0); x_0)$. Thus (50) holds for $p = q_{n,m,\theta_0,*}$.

Note that q_{n,m,θ_0} (and so also $q_{n,m,\theta_0,*}$) is really of degree $n-1$ when $\theta_0 = 0$, i.e., when $t = -1 + 2(m-1)/(n-1)$.

According to (24′), the sharp estimate for $p'(x)$ contained in (24) holds even if we *only* assume $p(t_{n,m}(x)) \leqslant 1$; i.e., $p((nx^2 + (n-2m+1)\,x - 1)/((n-1)\,x + n-2m+1)) \leqslant 1$.

3.2. *The Lower Bound for* $p'(x)$

In order to obtain the lower bound for $p'(x)$ it is sufficient to observe that the polynomial $p(-x)$ also belongs to π_n and so by (23) and (24),

$$-p'(-x) \leqslant \mathscr{E}_n(x) \qquad \text{for all} \quad x \in [-1, 1],$$

which is equivalent to (25).

3.3. *Proof of Corollary* 1

The case $n=1$ is covered by the result of Section 3.1.1. Let $n \geqslant 2$ and recall that the polynomials for which the pointwise bounds in Theorem 3 are attained have all their zeros on $\mathbb{R} \setminus (-1, 1)$. By (9) the estimate (6) holds at least for

$$|x| \leqslant \left\{1 - \frac{2}{\sqrt{e}\sqrt{n}} \left(1 - \frac{1}{n}\right)^{n-1}\right\}^{1/2}$$

and so, certainly, for $|x| \leqslant 3/4$.

Now note that, according to cases $m=1$ and $m=n$ of (23),

$$|p'(x)| \leqslant q'_{n,1,*}(x) \qquad \text{for} \quad -1 \leqslant x \leqslant \eta_{n,1} = -1 + \frac{3n - 1 - \sqrt{5n^2 - 6n + 1}}{n^2}$$

and so, definitely, for $-1 \leqslant x \leqslant -1 + 3/4n$. It follows that (6) holds for $1 - 3/4n \leqslant |x| \leqslant 1$ and, besides,

$$\left|p'\left(-1 + \frac{3}{4n}\right)\right| \leqslant q'_{n,1,*}\left(-1 + \frac{3}{4n}\right) \leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right)^{-n+1} \cdot n \cdot \frac{5}{8}. \tag{52}$$

Next, we observe that

$$1 + ax = \frac{1+a}{2}(1+x) + \frac{1-a}{2}(1-x)$$

and so, the polynomial $p(ax) \in \pi_n$ for all $a \in [-1, 1]$. Hence,

$$\left|p'\left(\left(-1 + \frac{3}{4n}\right) a\right)\right| \leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right)^{-n+1} \cdot n \cdot \frac{5}{8};$$

i.e.,

$$\left|p'\left(\left(-1 + \frac{3}{4n}\right) a\right)\right| \leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right)^{-n+1} \cdot n \cdot \frac{5}{8\,|a|} \qquad \text{for} \quad |a| \leqslant 1.$$

This implies that (6) holds for $\frac{5}{8} \leqslant |x| \leqslant 1 - 3/4n$.

3.4. *Proof of Corollary* 2

In view of (25), $-\mathscr{E}_n(0) \leqslant a_1 \leqslant \mathscr{E}_n(0)$. Hence, it is enough to determine $\mathscr{E}_n(0)$.

From (19) and (20), it readily follows that $0 \in [\xi_{n,m}, \eta_{n,m}]$ if and only if

$$\alpha_n := \frac{2n-1+\sqrt{4n+1}}{4} \leqslant m \leqslant \frac{2n+1+\sqrt{4n+1}}{4} =: \beta_n.$$

The following observations help us identify the values of n for which $[\alpha_n, \beta_n]$ contains an integer m:

I. $\beta_n - \alpha_n = \frac{1}{2}$ for all n and so α_n, β_n cannot both be integers.

II. $\alpha_{n+1} - \beta_n = 1/(\sqrt{4n+5} + \sqrt{4n+1}) < 1/(4\sqrt{n})$.

In view of I and II, at least one of the two intervals $[\alpha_{n-1}, \beta_{n-1}]$, $[\alpha_{n+1}, \beta_{n+1}]$ will contain an integer if $[\alpha_n, \beta_n]$ does not. Thus three consecutive intervals $\{[\alpha_\nu, \beta_\nu]\}_{\nu=n-1}^{n+1}$ cannot be devoid of integers.

III. Since

$$\begin{aligned}\beta_{n+2} - \alpha_n &< \beta_{n+2} - \alpha_{n+2} + \frac{1}{4\sqrt{n+1}} + \beta_{n+1} - \alpha_{n+1} + \frac{1}{4\sqrt{n}} + \beta_n - \alpha_n \\ &= \frac{3}{2} + \frac{1}{4\sqrt{n+1}} + \frac{1}{4\sqrt{n}} \\ &< 2 \qquad \text{for} \quad n \geqslant 1,\end{aligned}$$

the three intervals $[\alpha_n, \beta_n]$, $[\alpha_{n+1}, \beta_{n+1}]$, $[\alpha_{n+2}, \beta_{n+2}]$ cannot each contain an integer. However, two consecutive intervals can.

IV. For α_n or β_n to be an integer it is necessary and sufficient that $4n+1$ be the square of an integer. If α_n is an integer, then both $[\alpha_n, \beta_n]$, $[\alpha_{n+1}, \beta_{n+1}]$ contain an integer.

Since $4n+1 = (2k+1)^2$ if and only if $n = k(k+1)$ we consider the sequence $\{n_k\}_{k=1}^{\infty}$, where $n_k := k(k+1)$. For each k, either α_{n_k} or β_{n_k} is an integer. To be precise α_{n_k} $(=k(k+2)/2)$ is an integer if k is even and β_{n_k} $(=(k+1)^2/2)$ is an integer if k is odd. Note that the integer m_k lying in $[\alpha_{n_k}, \beta_{n_k}]$ can be written as $(n+k)/2$ or $(n+k+1)/2$, according as k is even or odd. Hence if $n = k(k+1)$ then a_1 is bounded above by $q'_{n,\,(n+k)/2,\,*}(0)$ if k is even and by $q'_{n,\,(n+k+1)/2,\,*}(0)$ if k is odd. Let k be even. Then $[\alpha_{n_k}, \beta_{n_k}]$, $[\alpha_{n_k+1}, \beta_{n_k+1}]$ both contain an integer and so do $[\alpha_{n_{k+1}-1}, \beta_{n_{k+1}-1}]$, $[\alpha_{n_{k+1}}, \beta_{n_{k+1}}]$. However, $[\alpha_{n_k+2}, \beta_{n_k+2}]$ does not. In general, $[\alpha_{n_k+2j}, \beta_{n_k+2j}]$ does not contain an integer as long as the fractional part of α_{n_k+2j}, i.e., the quantity

$$\tfrac{1}{4}\{\sqrt{4n_k+5}-\sqrt{4n_k+1}+\cdots+\sqrt{4n_k+8j+1}-\sqrt{4n_k+8j-3}\}$$
$$=\tfrac{1}{4}\{\sqrt{4k^2+4k+8j+1}-(2k+1)\}$$

is positive but less than $\frac{1}{2}$, i.e., for $1\leqslant j\leqslant k$. Note that if $n=n_k+2j-1$, where k is even and $1\leqslant j\leqslant k+1$ then the interval $[\alpha_n, \beta_n]$ contains the integer $m=m_k+j=k(k+2)/2+(n-n_k+1)/2=(n+k+1)/2$. Hence $a_1\leqslant q'_{n,(n+k+1)/2,*}(0)$ in this case.

In an analogous manner we see that if $n=n_k+2j$, where k is odd and $1\leqslant j\leqslant k$, then the interval $[\alpha_n, \beta_n]$ contains the integer $m=m_k+j=(k+1)^2/2+(n-n_k)/2=(n+k+1)/2$. Thus again $a_1\leqslant q'_{n,(n+k+1)/2,*}(0)$.

All values of n for which $[\alpha_n, \beta_n]$ contains an integer m have now been identified. The value of the integer m has also been determined which gives us the interval $[\xi_{n,m}, \eta_{n,m}]$ which contains the origin.

Now let us consider those n for which $[\alpha_n, \beta_n]$ does not contain an integer. We are assuming that $0\in(\eta_{n,m-1}, \xi_{n,m})$ for some m. The polynomial

$$p(x):=\frac{q_{n,m,0}(x)}{q_{n,m,0}(t_{n,m}(0))}=q_{n-1,m-1,*}(x)$$

maximizes a_1 if and only if $t_{n,m}(0)=-1+2(m-1)/(n-1)$. Since $t_{n,m}(0)=-1/(n-2m+1)$, this means that $n-1=j^2$, where $j=\pm(n-2m+1)$. Thus m is either equal to $(n+j+1)/2$ or to $(n-j+1)/2$. But the latter possibility is to be excluded since in that case $p'(0)<0$.

Now observe that $0\in(\eta_{n,m-1}, \xi_{n,m})$ if and only if

$$\frac{2n+1+\sqrt{4n+1}}{4}<m<\frac{2n+3+\sqrt{4n+1}}{4}. \tag{53}$$

If $n\in(n_k, n_{k+1})$ but does not fall in any of the preceding categories, then m must be equal to $(n+k+2)/2$. One way to see this is to note that $(n+k+2)/2$ is an integer and then to check that it lies in $((2n+1+\sqrt{4n+1})/4, (2n+3+\sqrt{4n+1})/4)$ which is an interval of length $\frac{1}{2}$. Now we refer to Step 5 in the proof of (24) and determine the (extremal) polynomial $q_{n,m,\theta_0,*}$ defined in (51).

With $m=(n+k+2)/2$, we obtain (see (21))

$$t_0=t_{n,m}(0)=\frac{1}{k+1}, \qquad \theta_0=\theta(0)=(k+1)\,\frac{(k+1)^2-(n-1)}{n-2(k+1)^2},$$

and

$$q_{n,m,\theta_0}(t_0) = \frac{(k+2)^{(n+k)/2}\, k^{(n-k-2)/2}}{(k+1)^{n-1}} \frac{(k+1)^2 - 1}{2(k+1)^2 - n}.$$

The polynomial $Q_{n,k,*}$ is nothing but $q_{n,m,\theta_0,*}$ of (51).

4. Proofs of Theorems 4, 5, and 5′

4.1. *Proof of Theorem* 4

It is easily seen that

$$q'_{n,k}(0) = -(n-2k), \qquad q''_{n,k}(0) = (n-2k)^2 - n,$$

and so for $\lambda \in \mathbb{R}$

$$\begin{aligned} q''_{n,k}(0) + \lambda q'_{n,k}(0) &= (n-2k)^2 - \lambda(n-2k) - n \\ &\geqslant (n-2k)^2 - |\lambda|\,|n-2k| - n \\ &= |n-2k|\,(|n-2k| - |\lambda|) - n. \end{aligned}$$

Now note that $|n-2k| \in \{0, 2, ..., n\}$ if n is even, whereas $|n-2k| \in \{1, 3, ..., n\}$ if n is odd and, hence,

$$|n-2k|\,(|n-2k| - |\lambda|) \geqslant \begin{cases} 0 & \text{for} \quad -2 \leqslant \lambda \leqslant 2 \qquad \text{if } n \text{ is even} \\ 1 - |\lambda| & \text{for} \quad -1 \leqslant \lambda \leqslant 1 \qquad \text{if } n \text{ is odd.} \end{cases}$$

It follows that if n is even then $q''_{n,k}(0) + \lambda q'_{n,k}(0) \geqslant -n$ for $-2 \leqslant \lambda \leqslant 2$ while if n is odd then $q''_{n,k}(0) + \lambda q'_{n,k}(0) \geqslant -n + 1 - |\lambda|$ for $-1 \leqslant \lambda \leqslant 1$. Since $p(x) := \sum_{k=0}^{n} A_k (1+x)^k\,(1-x)^{n-k}$, where $A_k \geqslant 0$ for all k, we obtain

$$\begin{aligned} p''(0) + \lambda p'(0) &= \sum_{k=0}^{n} A_k \{q''_{n,k}(0) + \lambda q'_{n,k}(0)\} \\ &\geqslant \min_{0 \leqslant k \leqslant n} \{q''_{n,k}(0) + \lambda q'_{n,k}(0)\} \cdot \sum_{k=0}^{n} A_k. \end{aligned}$$

Clearly $\sum_{k=0}^{n} A_k = p(0)$ and so if n is *even* then

$$p''(0) \geqslant -np(0) - \lambda p'(0)$$

for all λ in $[-2, 2]$, whence

$$p''(0) \geqslant -n + 2\,|p'(0)|. \tag{26′}$$

In order to complete the proof of Theorem 4 we observe that if n is *odd* then

$$p''(0) \geqslant (-n+1-|\lambda|)\, p(0) - \lambda p'(0) \qquad \text{for} \quad -1 \leqslant \lambda \leqslant 1;$$

i.e.,

$$p''(0) \geqslant -n+1-\delta+\delta\, |p'(0)| \qquad \text{for} \quad 0 \leqslant \delta \leqslant 1. \tag{26''}$$

4.2. *Proof of Theorem* 5

(i) Note that (30) cannot hold if $2\alpha \in (0, c]$. So we shall assume $\alpha > c/2$. Equating the right-hand sides of (29) and (30), we see that if $c \in (\sqrt{2}, c_n)$ and $\alpha > c/2$ satisfy the two equations, then

$$2\alpha = \Lambda_n(c). \tag{54}$$

Since $\Lambda_n(c)$ decreases to c_n on $[\sqrt{2}, c_n]$ we have

$$\frac{d}{dc}\left(\frac{2\alpha+c}{2\alpha-c}\right) = \frac{4}{(2\alpha-c)^2}\left(-c\,\frac{d\alpha}{dc}+\alpha\right) > 0$$

which implies that the right-hand side of (30) increases monotonically to ∞ as c increases to c_n. We have $\Lambda_4(\sqrt{2}) = 2.24891\ldots$, $\Lambda_5(\sqrt{2}) = 2.56804\ldots$; besides, the lower bound contained in (28) implies that $\Lambda_n(\sqrt{2}) > \sqrt{n/2}$ for all $n \geqslant 4$ and so

$$\left(\frac{1+\sqrt{2}/\sqrt{n}}{1-\sqrt{2}/\sqrt{n}}\right)^{\Lambda_n(\sqrt{2})\,\sqrt{n}} > \left(1+\frac{2\sqrt{2}}{\sqrt{n}}\right)^{n/\sqrt{2}} \geqslant 1+2\sqrt{n}+2(n-\sqrt{2})$$
$$> \frac{\Lambda_n(\sqrt{2})+\sqrt{2}}{\Lambda_n(\sqrt{2})-\sqrt{2}}.$$

In other words, the left-hand side of (30) is larger than its right-hand side at $c = \sqrt{2}$. Hence, the first part of Theorem 5 would follow if it could be shown that $g_n(c) := ((1+c/\sqrt{n})/(1-c/\sqrt{n}))^{\Lambda_n(c)\sqrt{n}}$ is a decreasing function of c on $[\sqrt{2}, c_n)$. Since

$$\log g_n(c) = \Lambda_n(c)\sqrt{n}\log\frac{\sqrt{n}+c}{\sqrt{n}-c} = \frac{\gamma^{3/2}}{c(\gamma-4)^{1/2}},$$

where $\gamma := \gamma_n(c)$, we see that

$$c^2(\gamma-4)\, g_n'(c) = g_n(c)\sqrt{\gamma/(\gamma-4)}\,\{c(\gamma-6)\,\gamma' - \gamma(\gamma-4)\}$$
$$= g_n(c)\sqrt{\gamma/(\gamma-4)}\left\{(\gamma-6)\left(\gamma+\frac{2c^2 n}{n-c^2}\right) - \gamma(\gamma-4)\right\}$$

$$= 2g_n(c)\sqrt{\gamma/(\gamma-4)}\,\frac{(c^2-1)\,n+c^2}{n-c^2}\left\{\gamma-\frac{6c^2}{(c^2-1)+c^2/n}\right\}$$

$$= g_n(c)\sqrt{\gamma/(\gamma-4)}\,\frac{4c^2}{n-c^2}\Bigg[\{(c^2-1)\,n+c^2\}$$

$$\times\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{c^2}{n}\right)^k-3n\Bigg]$$

$$<0$$

because $c<c_n<\sqrt{3-9/5n}$ and $\{(c^2-1)\,n+c^2\}\sum_{k=0}^{\infty}(1/(2k+1))(c^2/n)^k-3n$ (which is clearly an increasing function of c) is negative when $c=\sqrt{3-9/5n}$ as can be easily checked.

(ii) Let $f(x):=\int_0^n(1+x)^t\,(1-x)^{n-t}\,d\mu(t)\in\mathscr{F}_n$. It is easily seen that $f''(0)=\int_0^n\{(n-2t)^2-n\}\,d\mu(t)$ and so if we set $q_{n,t}(x):=(1+x)^t\,(1-x)^{n-t}$ then for all c in $[0,\sqrt{n})$ we have

$$f''(0)=\frac{1}{2}\int_0^n\frac{2\{(n-2t)^2-n\}}{q_{n,t}(c/\sqrt{n})+q_{n,t}(-c/\sqrt{n})}\left\{q_{n,t}\left(\frac{c}{\sqrt{n}}\right)+q_{n,t}\left(-\frac{c}{\sqrt{n}}\right)\right\}d\mu(t)$$

$$\leqslant\frac{1}{2}\left\{f\left(\frac{c}{\sqrt{n}}\right)+f\left(-\frac{c}{\sqrt{n}}\right)\right\}\max_{0\leqslant t\leqslant n}\frac{2\{(n-2t)^2-n\}}{q_{n,t}(c/\sqrt{n})+q_{n,t}(-c/\sqrt{n})}\tag{55}$$

$$=\left(1-\frac{c^2}{n}\right)^{-n/2}\frac{1}{2}\left\{f\left(\frac{c}{\sqrt{n}}\right)+f\left(-\frac{c}{\sqrt{n}}\right)\right\}\max_{0\leqslant t\leqslant n}\varphi_c(t),$$

where

$$\varphi_c(t):=\frac{2\{(n-2t)^2-n\}(1-c^2/n)^{(n-2t)/2}}{(1-c/\sqrt{n})^{n-2t}+(1+c/\sqrt{n})^{n-2t}}.$$

Since $\varphi_c(t)\leqslant 0$ if $|n-2t|\leqslant\sqrt{n}$ and $\varphi_c(t)\equiv\varphi_c(n-t)$ we indeed have

$$\max_{0\leqslant t\leqslant n}\varphi_c(t)=\max_{\sqrt{n}\leqslant n-2t\leqslant n}\varphi_c(t).$$

Note that in (55) equality holds for the function

$$f_c(x):=(1+x)^{t_c}\,(1-x)^{n-t_c}+(1+x)^{n-t_c}\,(1-x)^{t_c},$$

where $t_c\in[0,\frac{1}{2}(n-\sqrt{n})]$ is such that

$$\varphi_c(t_c)=\max_{\sqrt{n}\leqslant n-2t\leqslant n}\varphi_c(t).\tag{56}$$

Calculating the derivative of $\varphi_c(t)$ we see that the point $t = t_c$, where the desired maximum of φ_c is attained, satisfies

$$\left(\frac{1+c/\sqrt{n}}{1-c/\sqrt{n}}\right)^{n-2t} = \frac{\{(n-2t)^2 - n\} \log \dfrac{1+c/\sqrt{n}}{1-c/\sqrt{n}} + 4(n-2t)}{\{(n-2t)^2 - n\} \log \dfrac{1+c/\sqrt{n}}{1-c/\sqrt{n}} - 4(n-2t)}. \qquad (56')$$

No doubt, (55) gives us the best possible upper bound for $\sup_{f \in \mathscr{F}_n} f''(0)$ in terms of $\frac{1}{2}\{f(c/\sqrt{n}) + f(-c/\sqrt{n})\}$ for any prescribed c in $[0, \sqrt{n}]$; but if for some c the maximum of f_c on $[-1, 1]$ is attained at $\pm c/\sqrt{n}$ then for that c we will have

$$\frac{1}{2}\left\{f_c\left(\frac{c}{\sqrt{n}}\right) + f_c\left(-\frac{c}{\sqrt{n}}\right)\right\} = \max_{-1 \leqslant x \leqslant 1} f_c(x) \qquad (57)$$

and the corresponding bound for $\sup_{f \in \mathscr{F}_n} f''(0)$ given by (55) will also be the sharp upper bound for $\sup_{f \in \mathscr{F}_n} f''(0)$ in terms of $\max_{-1 \leqslant x \leqslant 1} f$. Differentiating f_c with respect to x and setting $x = c/\sqrt{n}$ we see that (57) is equivalent to

$$\left(\frac{1+c/\sqrt{n}}{1-c/\sqrt{n}}\right)^{n-2t} = \frac{n-2t+c\sqrt{n}}{n-2t-c\sqrt{n}} \qquad (t = t_c). \qquad (57')$$

The substitution $n - 2t = 2\alpha\sqrt{n}$ transforms (56′) into (29) and (57′) into (30). Applying part (i) of the theorem we conclude that there is a c in $(\sqrt{2}, c_n)$ and $t = t_c$ in $[0, \frac{1}{2}(n - c\sqrt{n})]$, satisfying (56′) and (57′) simultaneously. It is easily seen that with this choice of c inequality (55) reduces to (31).

4.3. *Proof of Theorem* 5′

(a) In view of (54), $c_{*,n}$ is a root of the equation

$$H_n(c) := \left(\frac{1+c/\sqrt{n}}{1-c/\sqrt{n}}\right)^{\Lambda_n(c)\sqrt{n}} - \frac{\Lambda_n(c)+c}{\Lambda_n(c)-c} = 0. \qquad (32')$$

Setting

$$\varepsilon_n = \varepsilon_n(c) := \sum_{k=1}^{\infty} \frac{1}{2k+1}\left(\frac{c^2}{n}\right)^k = \frac{c^2}{3n} + O\left(\frac{1}{n^2}\right),$$

we may write

$$\Lambda_n(c) = \frac{c}{\sqrt{c^2-2}} \sqrt{(1+\varepsilon_n)/[1 + (c^2/(c^2-2))\,\varepsilon_n]} \qquad (58)$$

and

$$\left(\frac{1+c/\sqrt{n}}{1-c/\sqrt{n}}\right)^{A_n(c)\sqrt{n}} = e^{2cA_n(c)(1+\varepsilon_n)} = \exp\left\{\frac{2c^2}{\sqrt{c^2-2}}\frac{(1+\varepsilon_n)^{3/2}}{(1+(c^2/(c^2-2))\,\varepsilon_n)^{1/2}}\right\}.$$

We note that for $\sqrt{2} < c \leqslant \sqrt{3}$,

$$(1+\varepsilon_n)^3 - \left(1+\frac{c^2}{c^2-2}\,\varepsilon_n\right)\left(1+\frac{3}{2}\,\varepsilon_n^2\right)^2 = -\frac{2(3-c^2)}{c^2-2}\,\varepsilon_n - \frac{2(c^2+1)}{c^2-2}\,\varepsilon_n^3 - \frac{9}{4}\,\varepsilon_n^4 - \frac{9c^2}{4(c^2-2)}\,\varepsilon_n^5 < 0$$

and, so,

$$\begin{aligned}\exp\left\{\frac{2c^2}{\sqrt{c^2-2}}\frac{(1+\varepsilon_n)^{3/2}}{(1+(c^2/(c^2-2))\,\varepsilon_n)^{1/2}}\right\} &< e^{2c^2/\sqrt{c^2-2}} \cdot e^{(3c^2/\sqrt{c^2-2})\,\varepsilon_n^2} \\ &= e^{2c^2/\sqrt{c^2-2}} + e^{2c^2/\sqrt{c^2-2}}(e^{(3c^2/\sqrt{c^2-2})\,\varepsilon_n^2} - 1).\end{aligned}$$

Further, for $\sqrt{(6+8\varepsilon_n)/3(1+\varepsilon_n)} < c \leqslant \sqrt{3}$ we have

$$\begin{aligned}&\left(1+\frac{c^2}{c^2-2}\,\varepsilon_n\right)\left(1-\frac{3-c^2}{c^2-2}\,\varepsilon_n\right)^2 - (1+\varepsilon_n)^3 \\ &\quad = -\frac{3}{(c^2-2)^2}\,\varepsilon_n^2 - \frac{3c^2-8}{(c^2-2)^3}\,\varepsilon_n^3 < 0,\end{aligned}$$

which means that

$$\begin{aligned}\exp\left\{\frac{2c^2}{\sqrt{c^2-2}}\frac{(1+\varepsilon_n)^{3/2}}{(1+(c^2/(c^2-2))\,\varepsilon_n)^{1/2}}\right\} &> e^{2c^2/\sqrt{c^2-2}} \cdot e^{-(2c^2(3-c^2)/(c^2-2)^{3/2})\,\varepsilon_n} \\ &> e^{2c^2/\sqrt{c^2-2}} - e^{2c^2/\sqrt{c^2-2}}\,\frac{2c^2(3-c^2)}{(c^2-2)^{3/2}}\,\varepsilon_n.\end{aligned}$$

Hence (for $\sqrt{(6+8\varepsilon_n)/3(1+\varepsilon_n)} < c \leqslant \sqrt{3}$),

$$\left(\frac{1+c/\sqrt{n}}{1-c/\sqrt{n}}\right)^{A_n(c)\sqrt{n}} = e^{2c^2/\sqrt{c^2-2}} + A(c,n)\,\varepsilon_n,$$

where

$$-e^{2c^2/\sqrt{c^2-2}}\,\frac{2c^2(3-c^2)}{(c^2-2)^{3/2}} < A(c,n) < e^{2c^2/\sqrt{c^2-2}}(e^{(3c^2/\sqrt{c^2-2})\,\varepsilon_n^2}-1). \quad (59)$$

On the other hand,

$$\begin{aligned}
&\frac{\Lambda_n(c)+c}{\Lambda_n(c)-c}\\
&\quad=\frac{\sqrt{1+\varepsilon_n}+\sqrt{c^2-2+c^2\varepsilon_n}}{\sqrt{1+\varepsilon_n}-\sqrt{c^2-2+c^2\varepsilon_n}}\\
&\quad=\frac{c^2-1+2\sqrt{c^2-2}\,\sqrt{1+2((c^2-1)/(c^2-2))\,\varepsilon_n+(c^2/(c^2-2))\,\varepsilon_n^2}+(c^2+1)\,\varepsilon_n}{3-c^2-(c^2-1)\,\varepsilon_n}
\end{aligned}$$

belongs to

$$\Bigg(\frac{c^2-1+2\sqrt{c^2-2}+(c+1)^2\,\varepsilon_n}{3-c^2-(c^2-1)\,\varepsilon_n},\quad \frac{c^2-1+2\sqrt{c^2-2}+\{(c^2+1)+2(c^2-1)/\sqrt{c^2-2}\}\,\varepsilon_n}{3-c^2-(c^2-1)\,\varepsilon_n}\Bigg)$$

because

$$\left(1+\frac{c}{\sqrt{c^2-2}}\,\varepsilon_n\right)^2 < 1+2\,\frac{c^2-1}{c^2-2}\,\varepsilon_n+\frac{c^2}{c^2-2}\,\varepsilon_n^2 < \left(1+\frac{c^2-1}{c^2-2}\,\varepsilon_n\right)^2.$$

Consequently,

$$\frac{\Lambda_n(c)+c}{\Lambda_n(c)-c}=\frac{1+\sqrt{c^2-2}}{1-\sqrt{c^2-2}}+B(c,n)\,\varepsilon_n,$$

where

$$\begin{aligned}
&\frac{(c^2-1)(1+\sqrt{c^2-2})^2+(3-c^2)(c+1)^2}{(3-c^2)\{3-c^2-(c^2-1)\,\varepsilon_n\}}\\
&\quad< B(c,n)\\
&\quad<\frac{(c^2-1)(1+\sqrt{c^2-2})^2+(3-c^2)\{(c^2+1)+2(c^2-1)/\sqrt{c^2-2}\}}{(3-c^2)\{3-c^2-(c^2-1)\,\varepsilon_n\}}.
\end{aligned} \quad (60)$$

Thus we have

$$H_n(c) = h(c) + A(c, n)\,\varepsilon_n - B(c, n)\,\varepsilon_n,$$

where $A(c, n)$ and $B(c, n)$ satisfy (59) and (60), respectively. Using the definition of c_* as the root of $h(c) = 0$ in $(\sqrt{2}, \sqrt{3})$, it is easily seen that

$$H_n(c_*) = A(c_*, n)\,\varepsilon_n - B(c_*, n)\,\varepsilon_n < 0$$

if $c_* < \theta_n := \sqrt{(3+\varepsilon_n(c))/(1+\varepsilon_n(c))} = \sqrt{3} - c^2/(3\sqrt{3}\,n) + O(1/n^2)$ (which implies that $3 - c_*^2 - (c_*^2 - 1)\,\varepsilon_n(c) > 0$) and that $H_n(c) < 0$ *immediately* to the left of θ_n if $\theta_n \leqslant c_*$. Hence, for *each* $n \geqslant 4$, there exist points to the left of c_*, where $H_n(c)$ is negative. While proving Theorem 5 we have shown that $H_n(\sqrt{2}) > 0$ and so the zero $c_{*,n}$ of H_n lies somewhere in $(\sqrt{2}, c_*)$; i.e., (a) holds.

(b) We shall now prove that if $n \geqslant 410$ then already $H_n(c_* - 1.35/(n-3))$ is positive; i.e.,

$$h(c) > (-A(c, n) + B(c, n))\,\varepsilon_n \qquad \text{at} \quad c = c_* - 1.35/(n-3).$$

For this we apply the mean value theorem to conclude that if $1.451 < c < c_*$, then for some ξ_c in (c, c_*)

$$h(c) = h(c_*) + h'(\xi_c)(c - c_*) = h'(\xi_c)(c - c_*).$$

Since

$$h'(\xi) = -2\xi\left(\frac{4-\xi^2}{(\xi^2-2)^{3/2}}\, e^{2\xi^2/\sqrt{\xi^2-2}} + \frac{1}{(1-\sqrt{\xi^2-2})^2\,\sqrt{\xi^2-2}}\right),$$

we conclude that

$$h(c) > \frac{2c}{(1-\sqrt{c^2-2})^2\,\sqrt{c^2-2}}\,|c - c_*|.$$

On the other hand, if

$$S(c, n) := (c^2-1)(1+\sqrt{c^2-2})^2 + (3-c^2)\left\{c^2+1+\frac{2(c^2-1)}{\sqrt{c^2-2}}\right\}$$
$$+ (3-c^2)^2\,\{3 - c^2 - (c^2-1)\,\varepsilon_n\}\,\frac{2c^2}{(c^2-2)^{3/2}}\, e^{2c^2/\sqrt{c^2-2}},$$

then

$$-A(c,n)+B(c,n)<\frac{1}{(3-c^2)\{3-c^2-(c^2-1)\,\varepsilon_n\}}\,S(c,n)$$
$$=\frac{c^2}{(c^2-2+c^2\varepsilon_n)(3-c^2)(\Lambda_n^2(c)-c^2)}\,S(c,n), \quad (61)$$

by (58). At this stage we need to find a lower estimate for $\Lambda_n^2(c)-c^2$. Since $\alpha=\frac{1}{2}\Lambda_n(c)$ satisfies (30) we have

$$\frac{1}{\Lambda_n^2(c)-c^2}=\frac{1}{(\Lambda_n(c)+c)^2}\exp\{2c(1+\varepsilon_n(c))\,\Lambda_n(c)\}. \quad (62)$$

Putting $t_n:=\Lambda_n^2(c)-c^2$, we find

$$\frac{1}{t_n}=\frac{1}{(\sqrt{t_n+c^2}+c)^2}\,e^{2c(1+\varepsilon_n(c))\sqrt{t_n+c^2}}$$
$$\leqslant\frac{1}{(\sqrt{t_n+c^2}+c)^2}\,e^{2c(1+\varepsilon_n(\sqrt{3}))\sqrt{t_n+c^2}}$$
$$\leqslant\frac{1}{(\sqrt{t_n+3}+\sqrt{3})^2}\,e^{2\sqrt{3}\,(1+\varepsilon_n(\sqrt{3}))\sqrt{t_n+3}}.$$

This implies that $t_n=\Lambda_n^2(c)-c^2$ cannot be smaller than the only positive root of the equation

$$(\sqrt{t+3}+\sqrt{3})^2=te^{2\sqrt{3}\,(1+\varepsilon_n(\sqrt{3}))\sqrt{t+3}} \quad (63)$$

which can be easily handled numerically. In particular, we obtain

$$t_4\geqslant 0.00130\ldots,\qquad t_{10}\geqslant 0.01402\ldots,\qquad t_{30}\geqslant 0.02355\ldots,$$
$$t_{100}\geqslant 0.02734515\ldots,\qquad t_{300}\geqslant 0.02846778\ldots.$$

The lower bound for t_n obtained in this manner is an increasing function of n since the root of (63) clearly is.

For the remainder of this proof we shall assume $n\geqslant 410$ so that $t_n>0.02846778\ldots$ and $1/(\Lambda_n^2(c)-c^2)<35.1275$. Returning to (61) we find that

$$(3-c^2)(-A(c,n)+B(c,n))$$
$$<35.1275\left[\frac{c^2(c^2-1)(1+\sqrt{c^2-2})^2}{c^2-2}+(3-c^2)\,\frac{c^2}{c^2-2}\right.$$
$$\left.\times\left\{c^2+1+\frac{2(c^2-1)}{\sqrt{c^2-2}}\right\}+(3-c^2)^3\,\frac{2c^4}{(c^2-2)^{5/2}}\,e^{2c^2/\sqrt{c^2-2}}\right].$$

Now we observe that $c^2(c^2-1)(1+\sqrt{c^2-2})^2/(c^2-2)$ is an increasing function of c on the interval $[1.59808..., \sqrt{3}]$ so that $c^2(c^2-1)(1+\sqrt{c^2-2})^2/(c^2-2) \leqslant 24$. It follows that if $c \in (c_* - \frac{1}{300}, c_*)$, then

$$(3-c^2)(-A(c,n)+B(c,n)) < (35.1275)(25);$$

i.e.,

$$(1-\sqrt{c^2-2})(-A(c,n)+B(c,n)) < 878.186/(1+\sqrt{c^2-2}).$$

Hence $H_n(c) > 0$, provided

$$\frac{-2c}{(1-\sqrt{c^2-2})\sqrt{c^2-2}}(c-c_*) > \frac{878.186}{1+\sqrt{c^2-2}}\varepsilon_n(c),$$

and so, certainly, if

$$\frac{1}{300} > -(c-c_*) > 439.093\left(\frac{(1-\sqrt{c^2-2})\sqrt{c^2-2}}{(1+\sqrt{c^2-2})\,c}\right)_{c=c_*-1/300} \cdot \varepsilon_n(c_*).$$

Both these inequalities are satisfied for $c = c_* - 1.35/(n-3)$ if $n \geqslant 410$. Hence (b) holds.

The bound in (31′) is based on an estimate for the quantity $U := \sqrt{4\alpha^2 - c^2}$ appearing in (31). Referring to (30) and (62) we see that U satisfies the equation

$$U = (\sqrt{U^2+c^2} - c)\, e^{c(1+\varepsilon_n(c))\sqrt{U^2+c^2}}$$

and so

$$U \leqslant (\sqrt{U^2+c_*^2} - c_*)\, e^{c_*(1+\varepsilon_n(c_*))\sqrt{U^2+c_*^2}}$$

$$= \frac{U^2}{\sqrt{U^2+c_*^2}+c_*}\, e^{c_*(1+\varepsilon_n(c_*))\sqrt{U^2+c_*^2}},$$

i.e.,

$$1 \leqslant \frac{U}{\sqrt{U^2+c_*^2}+c_*}\, e^{c_*(1+\varepsilon_n(c_*))\sqrt{U^2+c_*^2}}.$$

Hence, U does not exceed the only root U_n of the equation

$$1 = \frac{U}{\sqrt{U^2+c_*^2}+c_*}\, e^{c_*(1+\varepsilon_n(c_*))\sqrt{U^2+c_*^2}}. \tag{64}$$

It is easily checked that U_n increases if $\varepsilon_n(c_*)$ is replaced by something smaller. However, $\varepsilon_n(c_*) > 0$ for all n and so U is *always* smaller than the root $U_\infty := 0.171758633\ldots$ of the equation

$$1 = \frac{U}{\sqrt{U^2 + c_*^2} + c_*} \, e^{c_* \sqrt{U^2 + c_*^2}}. \tag{64'}$$

From the estimate $\sqrt{4\alpha^2 - c^2} \leqslant U_\infty$ it follows that $2\alpha \leqslant \sqrt{c_*^2 + U_\infty^2}$. This, in turn, gives an upper estimate for $(4\alpha^2 - 1)/2\alpha = 2\alpha - 1/2\alpha$, which when multiplied by U_∞ gives the number $0.199631037\ldots$ appearing in (31′).

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